IIT DHARWAD AUTUMN 2019

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Engg. Mathematics for Advanced Studies

MODULE - VECTOR CALCULUS

Assignment 02 - Solutions (Sample)

Q.1

$$\vec{r}(t) = a\cos(t)\cdot\hat{i} + b\sin(t)\cdot\hat{j}$$

velocity : $\vec{v} = \dfrac{d\vec{r}}{dt} = -a\sin(t)\cdot\hat{i} + b\cos(t)\cdot\hat{j}$

acceleration: $\vec{a} = \dfrac{d\vec{v}}{dt} = -a\cancel{\sin}\cos(t)\cdot\hat{i} - b\sin(t)\hat{j} = -\vec{r}(t)$

Q.2

$$\vec{f} = x^2\hat{i} + y^2\hat{k}$$

A) Direction ~~op~~ opposit to $\vec{f}$ will have most resistance.

$\therefore \vec{u} = -\vec{f} = -x^2\hat{i} - y^2\hat{k}$

B) direction of movement caused by force field

$\vec{v} = \vec{f} = x^2\hat{i} + y^2\hat{j}$

c) work done by force will be zero if displacement is perpendicular to it. Say $\vec{w} = f(x,y)\hat{i} + g(x,y)\hat{j}$ is $\perp^{er}$ to $\vec{f} \Rightarrow \vec{w}\cdot\vec{f} = 0$.

$\therefore x^2 f(x,y) + y^2 g(x,y) = 0$ —— (2c)

if $f(x,y) = -y^2$ & $g(x,y) = x^2$ eqn (2c) will be obeyed.

$\therefore \perp^{er}$ direction $\vec{w} = -y^2\hat{i} + x^2\hat{j}$ //

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$\boxed{Q.3}$

Find unit surface normal

a) $ax + by + cz + d = 0$ at $P(x_p, y_p, z_p)$

b) $x^2 + y^2 + 2z^2 = 26$ at $P(2,2,3)$

a) It is clearly a plane in 3D space.

normal to this plane is $a\hat{i} + b\hat{j} + c\hat{k}$

$\therefore$ unit surface normal $= \dfrac{a\hat{i} + b\hat{j} + c\hat{k}}{\sqrt{a^2+b^2+c^2}}$

b) $F(x,y,z) = x^2 + y^2 + 2z^2 - 26$

for $F(x,y,z)$ at (a,b,c), tangent plane is given by,

$$\frac{\partial F(a,b,c)}{\partial x}(x-a) + \frac{\partial F(a,b,c)}{\partial y}(y-b) + \frac{\partial F(a,b,c)}{\partial z}(z-c) = 0$$

Here, $(a,b,c) = (2,2,3)$ &

$$\frac{\partial F}{\partial x}(a,b,c) = 2x\Big|_{(a,b,c)} = 2a = 4$$

$$\frac{\partial F}{\partial y}(a,b,c) = 2y\Big|_{(a,b,c)} = 2b = 4$$

$$\frac{\partial F}{\partial z}(a,b,c) = 4z\Big|_{a,b,c} = 4c = 12$$

$$\therefore \quad 4(x-2) + 4(y-2) + 12(z-3) = 0$$

$$4x + 4y + 12z = 52$$

$$x + y + 3z = 13$$

$\therefore$ direction of the normal $\cancel{x+y}$ $(\hat{i} + \hat{j} + 3\hat{k})$

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$f = xyz$

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= yz\,\hat{i} + xz\,\hat{j} + xy\,\hat{k}$

At $P(-1, 1, 3)$,

$\nabla f\Big|_p = 3\hat{i} + (-3)\hat{j} - \hat{k}$

$\therefore D_v f = (3\hat{i} - 3\hat{j} - \hat{k})\cdot(\hat{i} - 2\hat{j} + 2\hat{k}) = 3 + 6 - 2 = +7$

$\therefore$ Directional derivative of $f = xyz$ at $P(-1,1,3) = 7$ along $\vec{v} = \hat{i} - 2\hat{j} + 2\hat{k}$

Q.5

$\vec{v} = y\,\hat{i}$ $\quad \vec{\nabla}\cdot\vec{v} = 0$ $\quad \left(\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z}\right.$

$= \cancel{\partial 0} + \frac{\partial(y)}{\partial x} + \frac{\partial(0)}{\partial y} + \left.\frac{\partial(0)}{\partial z}\right)$

$= 0$

Flow is incompressible

Q.6

$\vec{\nabla}\cdot\nabla f = \vec{\nabla}\cdot\left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right) = \vec{\nabla}\cdot\Big(2e^{2x}\sin(2y)\,\hat{i} + 2e^{2x}\cos(2y)\,\hat{j} + 0\Big)$

$= \vec{\nabla}\cdot\left(2e^{2x}\sin(2y)\,\hat{i} + 2e^{2x}\cos(2y)\hat{j}\right)$

$= \frac{\partial}{\partial x}\left(2e^{2x}\sin(2y)\right) + \frac{\partial}{\partial y}\left(2e^{2x}\cos(2y)\right)$

$= 4e^{2x}\sin(2y) - 4e^{2x}\sin(2y) = 0$

$\boxed{\vec{\nabla}\cdot\nabla f = 0}$

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Q.7

$$\vec{v} = x\hat{i} + y\hat{j} - z\hat{k}$$

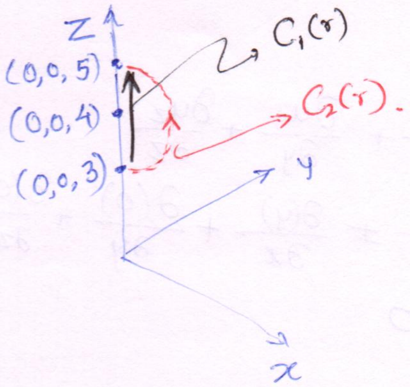
$$\vec{\nabla}\cdot\vec{v} = \frac{\partial(x)}{\partial x} + \frac{\partial(y)}{\partial y} + \frac{\partial(-z)}{\partial z} = 1 + 1 - 1 = 1$$

$\therefore$ As $\vec{\nabla}\cdot\vec{v} \neq 0 \Rightarrow$ Flow is NOT incompressible

$$\vec{\nabla}\times\vec{v} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & -z \end{vmatrix} = 0 \Rightarrow \boxed{\text{Flow is irrotation.}}$$

Q.8

$$\vec{F}(r) = 5z\,\hat{i} + xy\,\hat{j} + x^2 z\,\hat{k}$$



(a) $C_1 \rightarrow$ Straight line segment

$$r_1(t) = (0, 0, 3+2t) \qquad 0 \leq t \leq 1$$

(b) $C_2 \rightarrow$ Semicircle in x-z plane

$$r_2(t) = (\cancel{1}\sin(\pi t), 0, 4 - \cos(\pi t)) \qquad 0 \leq t \leq 1$$

(a) $$\int_{C_1} F(r)\cdot dr = \int_0^1 F(r_1(t)) \cdot r_1'(t)\,dt$$

$$= \int_0^1 \left(5(3+2t)\hat{i} + 0\hat{j} + 0\hat{k}\right)\cdot\left(0\hat{i} + 0\hat{j} + 2\hat{k}\right) dt$$

$$= \int_0^1 0\,dt = 0.$$

$$\boxed{\int_{C_1} \vec{F}(r)\cdot d\vec{r} = 0}$$

$\boxed{Q.8}$ contnd.

(b) $$\int_{C_2} \vec{F}(r)\cdot d\vec{r} = \int_0^1 \left(5\,(4-\cos(\pi t))\,\hat{i} + \sin^2(\pi t)\cdot(4-\cos(\pi t))\,\hat{k} \right) \cdot \left(\pi\cos(\pi t)\,\hat{i} + \pi\sin(\pi t)\,\hat{k}\right)$$

$$\cancel{= \int_0^1 \left(\pi\cos(\pi t)\,\hat{i} + \pi\sin(\pi t)\,\hat{j}\right)}$$

$$= \int_0^1 \left(5\pi\,(4\cos(\pi t) - \cos^2(\pi t))\,\hat{i} + \pi\sin^3(\pi t)(4-\cos(\pi t))\,\hat{k} \right) dt$$

$$= \int_0^1 \left(20\pi\cos(\pi t) - 5\pi\cos^2(\pi t) + 4\pi\sin^3(\pi t)\;\cancel{4} - \cancel{4}\pi\sin^3(\pi t)\cdot\cos(\pi t) \right) dt$$

$$= \int_0^\pi 20\cos\theta\cdot d\theta - \frac{5}{2}\int_0^\pi 2\cos^2\theta\cdot d\theta + 4\int_0^\pi \sin^3\theta\cdot d\theta$$

$$- \int_0^\pi \sin^3\theta\cdot\cos\theta\, d\theta \quad \ldots$$ (put $\theta = \pi t$. $\therefore d\theta = \pi dt$

$$= \underbrace{\left[20\sin\theta\right]_0^\pi}_{=0} - \frac{5}{2}\int_0^\pi (\cos 2\theta + 1)\,d\theta + 4\int_0^\pi \sin^3\theta\cdot d\theta$$

$$- \underbrace{\int_0^0 y^3\cdot dy}_{=0} \quad \ldots$$ (putting $y = \sin\theta$
$\therefore dy = \cos\theta\cdot d\theta$
when $\theta = 0 \Rightarrow y = 0$
$\theta = \pi \Rightarrow y = 0$

$$= -\frac{5}{2}\cdot\frac{1}{2}\underbrace{\left[\sin 2\theta\right]_0^\pi}_{=0} - \frac{5}{2}\left[\theta\right]_0^\pi + 4\int_0^\pi \sin^3\theta\, d\theta$$

$$= -\frac{5\pi}{2} + 4\int_0^\pi \sin^3\theta\, d\theta = \frac{-5\pi}{2} + 4\int_0^\pi (1-\cos^2\theta)\sin\theta\cdot d\theta$$

$$= -\frac{5\pi}{2} + 4\left[-\cos(\theta) + \frac{1}{3}\cos^3(\theta)\right]_0^\pi$$

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$\boxed{Q.8}$ contnd...

$$= -\frac{5\pi}{2} + 4\left[2 - \frac{2}{3}\right] = \frac{32 - 15\pi}{6}$$

$$\therefore \boxed{\int_{C_2} F(r) \cdot dr = \frac{32 - 15\pi}{6}}$$

$\boxed{Q.9}$

$$\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_R \vec{\nabla} \cdot \vec{F}\, dV$$

Given $\vec{F} = x^3 \hat{i} + y^3 \hat{j} + z^3 \hat{k}$

$$\therefore \vec{\nabla} \cdot \vec{F} = 3x^2 + 3y^2 + 3z^2$$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, dA = \cancel{3}\iiint_R \vec{\nabla} \cdot \vec{F}\, dV = 3 \iiint_R (x^2 + y^2 + z^2)\, dV$$

~~Notke~~ Note that the region 'R' for volume integration is a sphere of radius '3' with center at $(0,0,0)$. If this volume is considered as collection of spherical surfaces of radius 'r' going from 0 to $R = 3$, above function $(x^2 + y^2 + z^2)$, say $\phi(r) = r^2 = x^2 + y^2 + z^2$ is constant across surface of the sphere. ~~As area~~ It can be taken outside surface integral & integral along radial direction can proceed. Hence,

$$\iint_S \vec{F} \cdot \hat{n}\, dA = 3\int_0^3 r^2 \cdot (4\pi r^2) \cdot dr = 12\pi \int_0^3 r^4\, dr = 12\pi \left(\frac{r^5}{5}\right)_0^3$$

$$= \frac{12\pi}{5}\left[3^5 - 0^5\right] = \frac{2916\pi}{5}$$