IIT DHARWAD AUTUMN 2019

pg01
A01 - soln.
Vec. Calc.
Au19

Engg. Mathematics for Advanced Studies

MODULE - Vector Calculus

Solutions - Assignment 01

Q.1

(a) Vector form: $\vec{p} + t\vec{v} = (2\hat{i} + 3\hat{j} + 4\hat{k}) + t(2\hat{i} + \hat{j} + 2\hat{k})$

$-\infty < t < \infty$

(b) Parametric form:

$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$

with $(x_0, y_0, z_0) = (2, 3, 4)$ & $(a, b, c) = (2, 1, 2)$

$\therefore x = 2 + 2t \;;\; y = 3 + t \;;\; z = 4 + 2t$

(c) Symmetric form:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

$$\therefore \frac{x-2}{2} = \frac{y-3}{1} = \frac{z-4}{2}$$

(d) $\vec{r_q} = \vec{p} + 4\left(\frac{1}{\sqrt{9}}(2\hat{i} + \hat{j} + 2\hat{k})\right)$

$= (2\hat{i} + 3\hat{j} + 4\hat{k}) + \frac{4}{3}(2\hat{i} + \hat{j} + 2\hat{k})$

$\vec{r_q} = \frac{14}{3}\hat{i} + \frac{13}{3}\hat{j} + \frac{20}{3}\hat{k}$

$\therefore \|\vec{r_q}\| = \frac{1}{3}\sqrt{14^2 + 13^2 + 20^2} = \frac{1}{3}\sqrt{196 + 169 + 400} = \frac{\sqrt{765}}{3}$

$\therefore \vec{r}_p = 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\therefore \|\vec{r}_p\| = \sqrt{29}$

$\therefore \vec{r}_p \cdot \vec{r}_q = \left(\frac{28}{3} + \frac{39}{3} + \frac{80}{3}\right) = 49 \Rightarrow \cos\theta = \frac{\vec{r}_p \cdot \vec{r}_q}{\|\vec{r}_p\|\|\vec{r}_q\|} = \frac{49}{\sqrt{29}\left(\frac{\sqrt{765}}{3}\right)}$

Q.1

$$\cos\theta = \frac{147}{\sqrt{29\times 765}} \Rightarrow \theta = \cos^{-1}\left(\frac{147}{\sqrt{22185}}\right)$$

$$\theta = \cos^{-1}\left(\frac{147}{148.946}\right) = \cos^{-1}(0.98693)$$

$$\boxed{\theta = \cos^{-1}\left(\frac{147}{\sqrt{29\times765}}\right) = \cos^{-1}\left(\frac{147}{\sqrt{22185}}\right) = \cos^{-1}(0.98693)}$$

Q.2

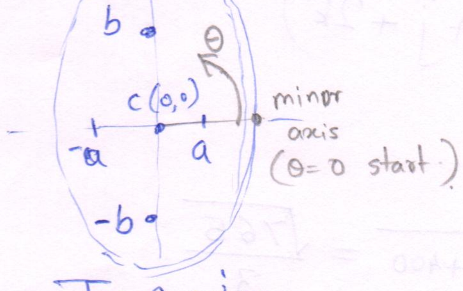
For any ellipse (~~on~~ in single plane)

$$\vec{r} = a\cos\theta\,\hat{i} + b\sin\theta\,\hat{j}$$

Here, the curve being an elliptical spiral with 40 cm pitch per turn around the pillar i.e. $\frac{40}{2\pi}$ cm per rad pitch

ⓐ $$\boxed{\vec{r}(\theta) = a\cos\theta\,\hat{i} + b\sin\theta\,\hat{j} + \frac{40}{2\pi}\theta.\hat{k}}$$

where, θ is total angular displacement as one traverses along the length of the light string



Top view

with $a = 20$ & $b = 25$ cm.

$$\boxed{\vec{r}(\theta) = 20\cos\theta\,\hat{i} + 25\sin\theta\,\hat{j} + \frac{40}{2\pi}\theta\,\hat{k}}$$

ⓑ 40 cm per turn $\Rightarrow$ Total 12.5 turns as height is 500 cm.

$$\boxed{\therefore\ \theta \text{ range} \Rightarrow 0 \leq \theta \leq 25\pi}$$

Q.2

pg03
A01
Vec Calc.
Au19

ⓒ $\vec{r}(\theta) = 20\cos\theta\,\hat{i} + 25\sin\theta\,\hat{j} + \frac{40}{2\pi}\theta\,\hat{k}$ — (2b)

$\vec{r}'(\theta) = -20\sin\theta\,\hat{i} + 25\cos\theta\,\hat{j} + \frac{40}{2\pi}\,\hat{k}$ — (2c)

Arc length required with $0 \leq \theta \leq 25\pi$

$$S = \int_0^{25\pi} \sqrt{r' \cdot r'}\; d\theta \quad \ldots \left(r' = \frac{d\vec{r}}{d\theta}\right).$$

$$= \int_0^{25\pi} \sqrt{(20\sin\theta)^2 + (25\cos\theta)^2 + \left(\frac{40}{2\pi}\right)^2}\; d\theta$$

$$= \int_0^{25\pi} \sqrt{400\sin^2\theta + 625\cos^2\theta + \frac{1600}{\cancel{16}\,4\,\pi^2}}\; d\theta$$

$$S = \int_0^{25\pi} \sqrt{400\sin^2\theta + 400\cos^2\theta + 225\cos^2\theta + \frac{400}{\pi^2}}\; d\theta$$

$$\boxed{S = \int_0^{25\pi} \sqrt{400 + 225\cos^2\theta + \frac{400}{\pi^2}}\; . \, d\theta}$$ — (2d)

ⓓ Using (2c),

$\vec{r}'(\theta)\Big|_{\theta=0} = 25\hat{j} + \frac{40}{2\pi}\hat{k}$ — minor axis tangent

$\vec{r}'(\theta)\Big|_{\theta=\frac{\pi}{2}} = -20\hat{i} + \frac{40}{2\pi}\hat{k}$ — major axis tangent.

Answers to part ⓔ ⓕ ⓖ ⓗ require help of numerical methods. Hence, those are not given any weightage for grading. Only (a) – (d) will carry all 8 marks

P904

Curvature:

$K(s) = \| r''(s) \|.$

Torsion $\tau(s) = -\vec{p}(s) \cdot b'(s)$

where, $\vec{b}'$ is binormal vector & $\vec{p}$ is principal normal

$$\frac{dz}{d\theta} = \frac{40}{2\pi}$$