IIT DHARWAD.
FALL 2019
Engg. Maths for Adv. Studies.

PDE Assignment 02 - Model Solutions

Q.1

(a) $\nabla^2 u = 0 \Rightarrow$ Laplace Equation.

(b) Spatial coordinates are typically the independent variables in Laplace Eqn.

(c) $\nabla^2 u$ in spherical coordinate system:
for $u = u(r, \theta, \phi)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{\cot\phi}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

(d) $\nabla^2 u$ in cyllindrical coordinate system:

for $u = u(r, \theta, z)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2}$$

(e) $\nabla^2 u$ in polar

for $u = u(r, \theta)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}$$

(e) $\nabla^2 u$ in cartesian

$u = \cancel{u(x, y, z)}.\ u(t, v)$

$$\nabla^2 u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial v^2}$$

Q.2

(a) $$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \quad \text{--- (1)}$$

For any $f = f(x^*)$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x^*} \cdot \frac{\partial x^*}{\partial x} + \frac{\partial f}{\partial y^*} \frac{\partial y^*}{\partial x} \quad \cdots \quad x = x(x^*, y^*)$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x^*} \frac{\partial x^*}{\partial y} + \frac{\partial f}{\partial y^*} \frac{\partial y^*}{\partial y} \quad \cdots \quad y = y(x^*, y^*)$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x^*}(1) + \frac{\partial f}{\partial y^*}(0) = \frac{\partial f}{\partial x^*}$$

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x^*}(0) + \frac{\partial f}{\partial y^*} \cdot (1) = \frac{\partial f}{\partial y^*}$$

Because,

$$x^* = x + a \Rightarrow \frac{\partial x^*}{\partial x} = 1, \ \frac{\partial x^*}{\partial y} = 0$$

$$y^* = y + b \Rightarrow \frac{\partial y^*}{\partial x} = 0, \ \frac{\partial y^*}{\partial y} = 1$$

$\therefore$ Going back to (1) & using this learning that

$$\frac{\partial}{\partial x}(f) = \frac{\partial}{\partial x^*}(f) \quad \& \quad \frac{\partial}{\partial y}(f) = \frac{\partial}{\partial y^*}(f)$$

$$\nabla^2 u = \frac{\partial}{\partial x^*}\left(\frac{\partial}{\partial x^*}(u)\right) + \frac{\partial}{\partial y^*}\left(\frac{\partial}{\partial y^*}(u)\right)$$

$$= \frac{\partial^2 u}{\partial x^{*2}} + \frac{\partial^2 u}{\partial y^{*2}} = \nabla_*^2 u$$

(b) if $$x^* = x\cos(\alpha) - y\sin(\alpha)$$

$$y^* = x\sin(\alpha) + y\cos(\alpha).$$

$$\frac{\partial x^*}{\partial x} = \cos(\alpha) \quad ; \quad \frac{\partial x^*}{\partial y} = -\sin(\alpha)$$

$$\frac{\partial y^*}{\partial x} = \sin(\alpha) \quad ; \quad \frac{\partial y^*}{\partial y} = \cos(\alpha).$$

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x^*}\frac{\partial x^*}{\partial x} + \frac{\partial f}{\partial y^*}\frac{\partial y^*}{\partial x}$$

$$= \cos(\alpha)\frac{\partial f}{\partial x^*} + \sin(\alpha)\frac{\partial f}{\partial y^*}$$

$$\frac{\partial^2 f}{\partial x^2} = \cos(\alpha)\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x^*}\right) + \sin(\alpha)\cdot\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y^*}\right)$$

$$= \cos(\alpha)\left[\frac{\partial}{\partial x^*}\left(\frac{\partial f}{\partial x^*}\right)\frac{\partial x^*}{\partial x} + \frac{\partial}{\partial y^*}\left(\frac{\partial f}{\partial x^*}\right)\frac{\partial y^*}{\partial x}\right]$$

$$+ \sin(\alpha)\left[\frac{\partial}{\partial x^*}\left(\frac{\partial f}{\partial y^*}\right)\frac{\partial x^*}{\partial x} + \frac{\partial}{\partial y^*}\left(\frac{\partial f}{\partial y^*}\right)\cdot\frac{\partial y^*}{\partial x}\right]$$

$$= \cos(\alpha)\left[\cos(\alpha)\frac{\partial^2 f}{\partial x^{*2}} + \sin(\alpha)\cdot\frac{\partial^2 f}{\partial y^*\partial x^*}\right]$$

$$+ \sin(\alpha)\left[\cos(\alpha)\cdot\frac{\partial^2 f}{\partial x^*\partial y^*} + \sin(\alpha)\frac{\partial^2 f}{\partial y^{*2}}\right]$$

$$\frac{\partial^2 f}{\partial x^2} = \cos^2(\alpha)\cdot\frac{\partial^2 f}{\partial x^{*2}} + 2\sin(\alpha)\cdot\cos(\alpha)\cdot\frac{\partial^2 f}{\partial x^*\partial y^*} + \sin^2(\alpha)\frac{\partial^2 f}{\partial y^{*2}}$$

$$\cdots\left(\because \frac{\partial^2 f}{\partial x^*\partial y^*} = \frac{\partial^2 f}{\partial y^*\partial x^*}\right)$$

(b1)

Now similarly.

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x^*}\cdot\frac{\partial x^*}{\partial y} + \frac{\partial f}{\partial y^*}\frac{\partial y^*}{\partial y}$$

$$= -\sin(\alpha)\cdot\frac{\partial f}{\partial x^*} + \cos(\alpha)\cdot\frac{\partial f}{\partial y^*}$$

$$\frac{\partial f}{\partial y} = -\sin(\alpha)\frac{\partial f}{\partial x^*} + \cos(\alpha)\frac{\partial f}{\partial y^*}$$

$$\therefore \frac{\partial^2 f}{\partial y^2} = -\sin(\alpha)\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x^*}\right)\right] + \cos(\alpha)\left[\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y^*}\right)\right]$$

$$= -\sin(\alpha)\left[\frac{\partial}{\partial x^*}\left(\frac{\partial f}{\partial x^*}\right)\cdot\frac{\partial x^*}{\partial y} + \frac{\partial}{\partial y^*}\left(\frac{\partial f}{\partial x^*}\right)\cdot\frac{\partial y^*}{\partial y}\right]$$

$$+ \cos(\alpha)\left[\frac{\partial}{\partial x^*}\left(\frac{\partial f}{\partial y^*}\right)\cdot\frac{\partial x^*}{\partial y} + \frac{\partial}{\partial y^*}\left(\frac{\partial f}{\partial y^*}\right)\frac{\partial y^*}{\partial y}\right]$$

$$= -\sin(\alpha)\left[-\sin(\alpha)\cdot\frac{\partial^2 f}{\partial x^{*2}} + \cos(\alpha)\cdot\frac{\partial^2 f}{\partial y^*\partial x^*}\right]$$

$$+ \cos(\alpha)\left[-\sin(\alpha)\cdot\frac{\partial^2 f}{\partial x^*\partial y^*} + \cos(\alpha)\cdot\frac{\partial^2 f}{\partial y^{*2}}\right]$$

$$= \sin^2(\alpha)\frac{\partial^2 f}{\partial x^{*2}} - 2\sin(\alpha)\cos(\alpha)\cdot\frac{\partial^2 f}{\partial x^*\partial y^*} + \cos^2(\alpha)\frac{\partial^2 f}{\partial y^{*2}} \quad \text{— (b2)}$$

from (b1) & (b2)

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \left(\cos^2(\alpha) + \sin^2(\alpha)\right)\frac{\partial^2 f}{\partial x^{*2}} + \left(2\sin(\alpha)\cos(\alpha) - 2\sin(\alpha)\cos(\alpha)\right)\frac{\partial^2 f}{\partial x^*\partial y^*}$$

$$+ \left(\cos^2(\alpha) + \sin^2(\alpha)\right)\frac{\partial^2 f}{\partial y^{*2}}$$

$$= \frac{\partial^2 f}{\partial x^{*2}} + \frac{\partial^2 f}{\partial y^{*2}}$$