IIT DHARWAD.

ENGG. MATHEMATICS FOR ADV. STUDIES | Fall 2019

Model Answers- PDE Assignmt 01

Q.1

ⓐ addition term Ax term is affecting the acceleration term $\left(\frac{\partial^2 u}{\partial t^2}\right)$ on the LHS. It is very much natural to to see 'Ax' as a force — Force that is proportional to spatial variable x linearly. Linear spring would contribute such force.

ⓑ $u_{tt} = c^2 u_{xx} + Ax$

if $u = F(x) \cdot G(t)$

$u_{tt} = F(x) \cdot \ddot{G}$ & $u_{xx} = F'' G(t)$.

$\therefore \quad F\ddot{G} = c^2 F'' G + Ax$

$$\frac{\ddot{G}}{G} = \frac{c^2 F''}{F} + \frac{Ax}{FG}$$ $\Rightarrow$ Not separable in terms of x & t terms.

ⓒ If $u(x,t) = y(x,t) + \psi(x)$

$u_x = \frac{dy(x,t)}{dx} + \psi'(x)$ | $u_t = \frac{dy(x,t)}{dt}$

$u_{xx} = \frac{d^2y(x,t)}{dx^2} + \psi''(x)$ | $u_{tt} = \frac{d^2y(x,t)}{dt^2}$

$u_{tt} = c^2 u_{xx}$ becomes

$$\frac{d^2y(x,t)}{dt^2} = c^2\left[\frac{d^2y(x,t)}{dx^2}\right] + c^2\psi''(x) + Ax$$

(d) $$\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2} + c^2\psi''(x) + Ax$$

can reduce to standard form $\frac{d^2y}{dt^2} = c^2\frac{d^2y}{dx^2}$

if $c^2\psi''(x) = -Ax$

i.e. $$\psi'(x) = \int \frac{-Ax}{c^2}\,dx + \cancel{\text{Const}}\ K_1$$

$$= -\frac{A}{c^2}\frac{x^2}{2} + K_1$$

integrating again.

$$\psi(x) = -\frac{A}{6c^2}x^3 + \cancel{A}k_1 x + k_2$$

K_1 & K_2 are constants of integration

(e) $$u(x,t) = y(x,t) + \psi(x).$$

$$u(x,t) = y(x,t) - \frac{A}{6c^2}x^3 + k_1 x + k_2$$

Given B.C. $u(0,t) = 0 \Rightarrow y(0,t) + \psi(0) = 0$

if $\psi(0) = 0$ then $y(0,t) = 0$.

To achieve that $k_2 = 0$.

Also $u(L,t) = 0 \Rightarrow y(L,t) + \psi(L) = 0$

To get $u(L,t) = y(L,t) \Rightarrow \psi(L) = 0 \Rightarrow \frac{-A}{6c^2}L^3 + k_1 L = 0$

$$\therefore K_1 = \frac{AL^2}{6c^2}$$

$$\therefore \psi(x) = -\frac{A}{6c^2}x^3 + \frac{AL^2}{6c^2}x + 0$$

$$\psi(x) = \frac{A}{6c^2}\left[xL^2 - x^3\right]$$

Note $\rightarrow$ $\psi(x)$ is such that $y(0,t) = y(L,t) = 0$

(f)
$$y(x,t) = u(x,t) - \psi(x).$$

(i)
$$\frac{d^2y}{dt^2} = c^2 \frac{d^2y}{dx^2}$$

(ii) Boundary conditions:
$$y(0,t) = y(L,t) = 0$$

(iii) Initial Conditions:

$u(x,0) = 0$ for $t > 0$ $\Rightarrow$ $y(x,0) = u(x,t) - \psi(x)$
$$= 0 - \frac{A}{6c^2}\left[xL^2 - x^3\right]$$
$$\therefore \; y(x,0) = -\frac{A}{6c^2}\left[xL^2 - x^3\right]$$

$\left.\frac{du}{dt}\right|_{(x,0)} = 1 \Rightarrow \frac{dy}{dt}(x,0) = \frac{du}{dt}(x,0) - 0$

for $0 < x < L$ $\qquad \frac{dy}{dt}(x,0) = 1$

$\boxed{Q.2}$

Given:
$$Z_{tt} = c^2\left(Z_{rr} + \frac{1}{r} Z_r\right)$$
$$Z(r,0) = f(r)$$
$$\frac{\partial z}{\partial t}(r,0) = g(r).$$

Let $z(r,t) = F(r) \cdot G(t) \Longrightarrow \frac{\partial z}{\partial r} = F'G \quad \frac{\partial z}{\partial t} = F\dot{G}$

$$\frac{\partial^2 z}{\partial r^2} = F''G \quad \frac{\partial^2 z}{\partial t^2} = F\ddot{G}$$

$$\therefore \quad Z_{tt} = c^2\left(Z_{rr} + \frac{1}{r}Z_r\right)$$

Becomes.

$$F\ddot{G} = c^2\left(F''G + \frac{1}{r}F'G\right)$$

$$\therefore \quad \frac{\ddot{G}}{c^2 G} = \frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = k$$

LHS is function of independent variable t & RHS is function of another independent variable r.

Hence, we can write.

$$\frac{\ddot{G}}{c^2 G} = k \neq 0 \quad \& \quad \frac{F''}{F} + \frac{1}{r}\frac{F'}{F} = k \neq 0$$

$$\therefore \quad \boxed{\ddot{G} - kc^2 G = 0 \quad \& \quad F'' + \frac{F'}{r} - kF = 0}$$

Q.3

$$u_{xx} - 3u_{yy} + 2u_y + u - y = \text{const.}$$

$$u_{xx} - 3u_{yy} = f^n(u_y, u, y)$$

Hessian matrix std. eqn. for $u(x,y)$, form v: $au_{xx} + 2bu_{xy} + cu_{yy} = f^n(u_x, u_y, u, y, x)$

$a=1$, $b=0$, $c=-3$

$$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & -3 \end{vmatrix} = -3 < 0$$

$\begin{vmatrix} a & b \\ b & c \end{vmatrix} > 0 \Rightarrow$ ~~parabolic~~ elliptic

$\begin{vmatrix} a & b \\ b & c \end{vmatrix} = 0 \Rightarrow$ parabolic

$\begin{vmatrix} a & b \\ b & c \end{vmatrix} < 0 \Rightarrow$ Hyperbolic

Hence, $\boxed{\text{Given PDE is Hyperbolic}}$