IIT DHARWAD AUTUMN 2019-20

ENGG. MATHS FOR ADVANCED STUDIES

MODULE - ODE. (FINAL EXAM)

Q.1

		ORDER	DEGREE	Linear Non-Linear	Homogeneous Non-Homogeneous	Autonomous
1.	$(y'')^3 + e^y = x$	2	3	NL	NH	NO
2.	$y'' + 3yy' + y = 0$	2	1	NL	H	YES
3.	$y'' + (y')^2 + 4y = e^x$	2	1	NL	NH	NO
4.	$y'' + 3yy' + y = \sin(y)$	2	1	NL	NH	YES

Q.2

If $y(t)$ is amount of salt in the tank,

$\frac{y(t)}{1000}$ salt per gallon is present

$\frac{50}{1000} \cdot y(t)$ i.e. $0.05\,y(t)$ salt outflows

$$\therefore \quad y'(t) = \text{inflow} - \text{outflow} = 50(1+\cos(t)) - 0.05\,y(t)$$

$$\boxed{y'(t) = 50 + 50\cos(t) - 0.05\,y(t)}$$

$$a(t) = -0.05$$

$$q(t) = \cancel{50(t} \; 50 + 50\cos(t).$$

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$\boxed{Q.3}$

Volume of sphere $= V = \frac{4}{3}\pi r^3$

Surface Area of sphere $= S = 4\pi r^2$

Given, volume loss is proportional to surface area.

$$\therefore \quad \frac{dV}{dt} = k \cdot S = 4\pi k r^2$$

However, $\frac{dV}{dt} = \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = \frac{4\pi}{3}(3r^2) \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\therefore \quad 4\pi r^2 \frac{dr}{dt} = 4\pi k r^2$$

$$\therefore \quad \frac{dr}{dt} = k$$

$$\therefore \quad r(t) = kt + C_0$$

At $t = 0$, $r_0 = 20$ mm. $\Rightarrow C_0 = r_0 = 20$

$t_1 = 60$, $r_1 = 10$ mm. $\Rightarrow 10 = k(60) + 20 \Rightarrow k = -\frac{1}{6}$

$$\therefore \quad r(t) = 20 - \frac{t}{6}$$

Hence, for $r_2 = 1$, $\quad 1 = 20 - \frac{t_2}{6} \Rightarrow t_2 = 19 \times 6 = 114$ days.

$\boxed{\text{Time to reach 1mm dia} = 114 \text{ days } (3.8 \text{ mōths})}$

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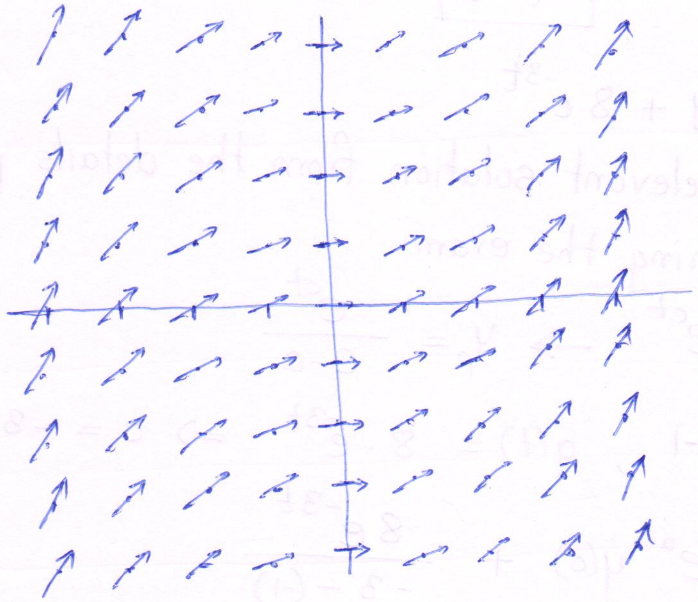
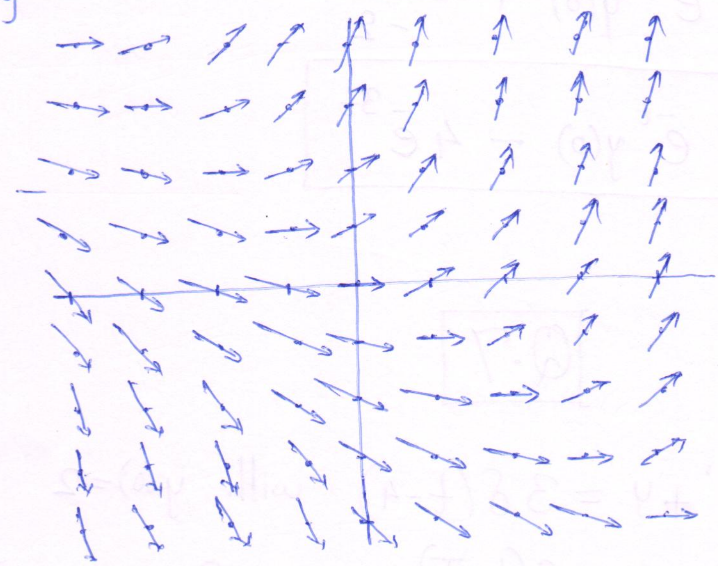
Q.4

a) $y' = x^2$



b) $y' = x + y$



Q.5

$$y' = ay + q(t) \Rightarrow y(t) = e^{at} \cdot y(0) + e^{at} \int_0^t e^{-as} \cdot q(s)\, ds$$

$$y' = -9y + 90 \Rightarrow a = -9,\ q = 90 \Rightarrow y(t) = e^{-9t} \cdot y(0) + e^{-9t} \int_0^t e^{+9s} \cdot (90)\, ds$$

$$\therefore y(t) = e^{-9t} \cdot y(0) + e^{-9t} \cdot (90) \left[\frac{1}{9} \cdot e^{9s} \right]_0^t$$

$$= e^{-9t} \cdot y(0) + 90 e^{-9t} \cdot \frac{1}{9} \left[e^{9t} - 1 \right]$$

$$= e^{-9t} y(0) + 10 \cancel{e^{-9}} \left(1 - e^{-9t} \right)$$

$$\boxed{y(t) = e^{-9t} y(0) + 10 \left(1 - e^{-9t} \right)}$$

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Q.6

$$y' = -y + 8e^{-3t}$$

Choosing the relevant solution from the details provided on the blackboard during the exam:

$$q(t) = e^{ct} \rightarrow y_p = \frac{e^{ct}}{c-a}$$

Here $a = -1$, $q(t) = 8 \cdot e^{-3t} \Rightarrow c = -3$

$$\therefore \quad y(t) = e^{at} y(0) + \frac{8e^{-3t}}{-3-(-1)}$$

$$= e^{-t} y(0) + \frac{8e^{-3t}}{-2}$$

$$\boxed{y(t) = e^{-t} y(0) - 4e^{-3t}}$$

Q.7

$$y' + y = 3\delta(t-4) \quad \text{with } y(0) = 2$$

$$y(t) = e^{at} y(0) + e^{a(t-T)} \quad \text{-- for } q = \delta(t-T)$$

Here, $y' = -y + 3\delta(t-4) \Rightarrow a = -1, \; q = 3\delta(t-4)$

$T = 4.$

$$\therefore \quad y(t) = e^{-t} \cdot y(0) + 3e^{-\cancel{3}(t-4)} \cdot H(t-4).$$

$$\boxed{y(t) = e^{-t} y(0) + 3e^{-(t-4)} \cdot H(t-4)}$$

Substituting $y(0) = 2$

$$\boxed{y(t) = 2e^{-t} + 3e^{-(t-4)} \cdot H(t-4)}$$

(Note - $H(t-4)$ is a step function which makes second term active only after $t \geq 4$)

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Q.8

$z' + z = \cancel{8}\, e^{8it}$

~~a = -1~~ Compare with $q(t) = Re^{i\omega t}$

$\omega = 8, \quad a = -1, \quad R = 1$

$y_n = y(0)\, e^{at} = y(0)\, e^{-t}$

$$y_p = \frac{Re^{i\omega t}}{i\omega - a} = \frac{e^{8it}}{8i+1} = \frac{e^{8it}}{1+8i}$$

Denominator $= 1 + 8i$

Convert it in polar format $re^{i\theta}$ where

$r = \sqrt{1^2 + 8^2} = \sqrt{65}$

$\theta = \tan^{-1}\left(\frac{8}{1}\right) = \tan^{-1}(8)$

$\therefore y_p = re^{i\theta} \cdot e^{8it} = re^{i(\theta + 8t)}$

$$\boxed{y_p = \sqrt{65}\, e^{i(\theta + 8t)} \quad \ldots \text{where } \theta = \tan^{-1}(8}$$

Q.9

$2xy - 9x^2 + (2y + x^2 + 1)\frac{dy}{dx} = 0$

$\therefore (9x^2 - 2xy)\,dx - (2y + x^2 + 1)\,dy = 0$

Comparing with $M(x,y)\,dx + N(x,y)\,dy = 0$

$M(x,y) = 9x^2 - 2xy \quad ; \quad N(x,y) = -(2y + x^2 + 1)$

$\frac{\partial M}{\partial y} = -2x \quad ; \quad \frac{\partial N}{\partial x} = -2x$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \implies$ Given Equation is Exact Differential Eqn

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Q.10

$$3\sin(4t) + 4\cos(4t) \quad \text{—— (10)a}$$

$$R\cos(\omega t - \phi) = R\left[\cos(\omega t)\cdot\cos(\phi) + \sin(\omega t)\cdot\sin\phi\right]$$

$$= (R\sin\phi)\sin(\omega t) + (R\cos\phi)\cos(\omega t). \quad \text{—— (10b)}$$

Comparing (10a) & (10b) $\quad \omega = 4$

$R\sin\phi = 3$ & $R\cos\phi = 4$.

$\therefore R = \sqrt{3^2+4^2} = 5$ & $\tan\phi = \frac{3}{4} \Rightarrow \phi = \tan^{-1}\left(\frac{3}{4}\right)$

$$\therefore \boxed{3\sin(4t) + 4\cos(4t) = 5\cos(4t - \phi) \quad \therefore \phi = \tan^{-1}\left(\frac{3}{4}\right)}$$

Q.11

$$\frac{dT}{dt} = k(T_{\infty} - T_{\infty})$$

$$\therefore \frac{dT}{T_{\infty} - T_{\infty}} = k\cdot dt$$

$$\therefore +\ln(T_{\infty} - T_{\infty}) = kt + \tilde{C}_0$$

$$\therefore T_{\infty} - T_{\infty} = e^{+kt+\tilde{C}_0} = e^{+\tilde{C}_0}\cdot e^{+kt}$$

$$\therefore T = T_{\infty} + \cancel{e}\, C_0 \cdot e^{+kt} \quad \cdots \left(C_0 = e^{\tilde{C}_0}\right)$$

At $t = 0$ $\quad T = 150$ & $T_{\infty} = 20 \Rightarrow 150 = 20 + C_0\cdot e^0 = 20 + C_0$

$\therefore C_0 = 130$.

At $t = 10$ $\quad T = 110$ & $T_{\infty} = 20 \Rightarrow 110 = 20 + 130e^{+k(10)}$

$$\therefore e^{10k} = \frac{90}{130}$$

$$\therefore k = -0.0367725$$

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Q.11

(b). $T = 37$ $T_\infty = 20$ $C_0 = 130$, $k = -0.036772$

$\therefore$ $T = T_\infty + C_0 e^{kt}$ becomes.

$$37 = 20 + 130\, e^{-0.036772t}$$

$$\therefore \quad -0.036772\, t = \ln\left(\frac{17}{130}\right)$$

$$t = 55.322 \text{ mins.}$$

$$\boxed{\text{Time to reach } 37^\circ C: \ t = 55.32 \text{ minutes}}$$

Q.12

$$y'' = 2y' + 35y.$$

$$\therefore \quad y'' - 2y' - 35y = 0$$

Assume soln. of format $y = Ce^{st}$ $\Rightarrow$ $y' = Cse^{st}$

$y'' = Cs^2e^{st}.$

substituting,

$$\therefore \quad s^2 - 2s - 35 = 0$$

$$s = 7 \quad \text{or} \quad s = -5$$

$$\therefore \quad \boxed{y = C_1 e^{7t} + C_2 e^{-5t}}$$

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Q.13

$y'' - y' - y = 0.$

$\therefore$ Similar to previous problem, characteristic eqn

$$s^2 - s - 1 = 0$$

$$\therefore \quad s = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{1+4}}{2}$$

$$s = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \quad \boxed{y = C_1 e^{\left(\frac{1+\sqrt{5}}{2}\right)t} + C_2 e^{\left(\frac{1-\sqrt{5}}{2}\right)t}}$$