IIT DHARWAD.

ENGG. MATHS FOR ADVANCED STUDIES

ODE - Assignment 02 - Solutions

Q.1

Standard form for: $y = y(t)$

i) Bernoulli Eqn. $\rightarrow$ $y' + p(t)y = q(t)\cdot y^a$

ii) Logistic Eqn. $\rightarrow$ $y' + ay = -by^2$

iii) Riccati Eqn $\rightarrow$ $y' + p(t)y = q(t)\cdot y^2 + h(t)$

Logistic eqn. is a special subset of Bernoulli Eqns, where coefficients are constants and the power of y is 2.

All 3 equations are non-linear.

Q.2

Riccati Eqn. $y' = x^3(y-x)^2 + \frac{y}{x}$

$w = y - x \Rightarrow w' = y' - 1 \Rightarrow y' = w' + 1$

$\Rightarrow \frac{w}{x} = \frac{y}{x} - 1 \Rightarrow \frac{y}{x} = \frac{w}{x} + 1$

$\therefore$ Given eqn. ,after above substitutions gives,

$$w' + 1 = x^3(w)^2 + \frac{w}{x} + 1$$

$$\therefore \quad w' - \frac{1}{x}w = x^3 w^2 \longrightarrow \text{Bernoulli Eqn Form}$$

power of w on RHS is 2. Hence, to get Bernoulli eqn. in linear form we ~~multiply~~ substitute

$$u = w^{1-2} = \frac{1}{w}.$$

$$u = \frac{1}{w} \Rightarrow w = \frac{1}{u} \Rightarrow w' = \frac{-u'}{u^2}$$

$$w' + \left(-\frac{1}{x}\right) w = x^3 w^2 \quad \text{becomes:}$$

$$\frac{-u'}{u^2} - \frac{1}{x} \cdot \frac{1}{u} = x^3 \frac{1}{u^2}$$

$$\therefore \quad u' + \frac{u}{x} = -x^3$$

Considering standard form $u' + p(x) \cdot u = q(x)$

$$p(x) = \frac{1}{x} \qquad q(x) = -x^3$$

$$\therefore \quad e^{\int p(x)\,dx} = e^{\int \frac{1}{x} dx} = e^{\ln x} = x \Rightarrow \text{integrating factor.}$$

Multiply throughout by x:

$$x u' + u = -x^4$$

$$\therefore \quad \frac{d}{dx}(xu) = -x^4$$

$$\therefore \quad xu = \int -x^4\,dx + C = \frac{-x^5}{5} + C$$

$$\therefore \quad u = \frac{5C - x^5}{5x}$$

$$\therefore \quad \frac{1}{w} = \frac{5C - x^5}{5x}$$

$$\therefore \quad \frac{1}{y - x} = \frac{5C - x^5}{5x}$$

$$\therefore \quad 5x = (y - x)(5C - x^5) = 5Cy - 5Cx - x^5 y + x^6$$

$$\therefore \quad x^6 - 5(C+1)x - (x^5 - 5C)\,y = 0.$$

$$\therefore \quad y = \frac{x^6 - 5(C+1)x}{x^5 - 5C} = \frac{x(x^5 - 5C) - 5x}{x^5 - 5C}$$

$$y = x - \frac{5x}{x^5 - 5C}$$

Q.3

$$\frac{dy}{dt} = \frac{g(t)}{f(y)}$$

$\therefore \quad g(t)\,dt - f(y)\,dy = 0$

$M = g(t) \qquad N = -f(y).$

$\frac{\partial M}{\partial y} = 0 \qquad \frac{\partial N}{\partial t} = 0 \implies \frac{\partial M}{\partial y} = \frac{\partial N}{\partial t} \implies$ Exactness is satisfied.

Q.4

ⓐ $my'' + ky = 0 \qquad y(0) = a \quad ; \; y'(0) = 0$

It is oscillatory motion, hence we can guess solution to be a sinusoidal one with some phase lag. i.e.

say $\quad y = R\cos(\omega_0 t - \phi).$

$\therefore \quad y'' = -\omega_0^2 R\cos(\omega_0 t - \phi).$

$\therefore \; my'' + ky = 0 \implies -m\omega_0^2 R\cos(\omega_0 t - \phi) + kR\cos(\omega_0 t - \phi) = 0$

$\therefore \quad R(k - m\omega_0^2)\cos(\omega_0 t - \phi) = 0$

$\therefore \quad k - m\omega_0^2 = 0 \implies \boxed{\omega_0 = \sqrt{\frac{k}{m}}}$

Also, $y(0) = a \implies R\cos(0 - \phi) = a \implies R\cos(-\phi) = a$

$y'(0) = 0 \implies -\omega_0 R\sin(0 - \phi) = 0 \implies \phi = 0.$

$\implies R = a.$

$\therefore$ Solution: $\boxed{y(t) = a\cos(\omega_0 t) \quad \text{with } \omega_0 = \sqrt{\frac{k}{m}}}$

Amplitude is constant. It will not shoot to infinity

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Q.4

(b) $$m y'' + ky = F_0 \cos(\omega t)$$

Take $y = a\cos\omega t + b\sin\omega t$ ... (Note that it has same periodicity as that of forcing function $F_0\cos(\omega t)$ i.e. ω. & also it is allowing phase difference as it contains both sin & cos).

$\therefore$ $y'' = -\omega^2 a\cos(\omega t) - \omega^2 b\sin(\omega t)$

$\therefore$ $my'' + ky = F_0\cos(\omega t)$ becomes:

$$-m\omega^2 a\cos(\omega t) - m\omega^2 b\sin(\omega t) + ka\cos(\omega t) + kb\sin(\omega t) = F_0\cos(\omega t)$$

$\therefore$ $(ka - m\omega^2 a)\cos(\omega t) + (kb - m\omega^2 b)\sin(\omega t) = F_0 \cos(\omega t)$

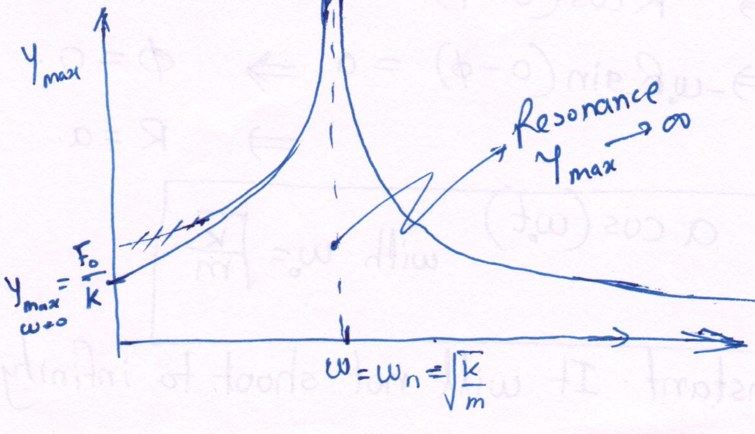
$\therefore$ Comparing coefficients of cos & sin terms:

$$\left.\begin{aligned} a(k - m\omega^2) &= F_0 \\ b(k - m\omega^2) &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= \frac{F_0}{k - m\omega^2} \\ b &= 0 \end{aligned}$$

$$\therefore \boxed{y = \frac{F_0}{k - m\omega^2}\cos(\omega t) \quad \text{i.e. amplitude } y_{max} = \left|\frac{F_0}{k - m\omega^2}\right|}$$

(Above derivation NOT expected for grading. Only the Plot given below is needed).

Hence, The plot below



Q.5

(a) $y = e^{-t} \Rightarrow y'' = (-e^{-t})' = e^{-t} = y \Rightarrow y'' - y = 0$

Hence $y = e^{-t}$ is solution.

(b) $y = e^{t} \Rightarrow y'' = (e^{t})' = e^{t} = y \Rightarrow y'' - y = 0$

Hence $y = e^{t}$ is a solution.

(c) $y = 4e^{t} - 3e^{-t}$ is also a solution

(d) Linear Homogeneous properties facilitate superimposition of two solutions to generate a new solution.

Q.6

(a) $4y'' \mp 4y' - 3y = 0 \qquad y'(-2) = -\frac{e}{2} \quad y(-2) = e.$

$4s^2 - 4s - 3 = 0 \quad ..($ if e^{st} is a canditate sol^n

$4s^2 e^{st} - 4se^{st} - 3e^{st} = 0$

$(4s^2 - 4s - 3)e^{st} = 0)$.

$4s^2 - 6s + 2s - 3 = 0$

$(2s+1)(2s-3) = 0$

$s = -\frac{1}{2}$, or $\frac{3}{2}$.

$\therefore \; y = C_1 e^{-\frac{t}{2}} + C_2 e^{\frac{3}{2}t} \Rightarrow y' = -\frac{C_1}{2} e^{-\frac{t}{2}} + \frac{3C_2}{2} e^{\frac{3}{2}t}$

Given $y'(-2) = -\frac{e}{2} \Rightarrow -\frac{e}{2} = -\frac{C_1}{2} e^{+1} + \frac{3C_2}{2} e^{-3}$

$\therefore \; C_1 e^4 - 3C_2 = e^4$

$y(-2) = e \Rightarrow e = C_1 e + C_2 e^{-3}$

$\therefore \; C_1 e^4 + C_2 = e^4$

$\therefore \; 4C_2 = 0 \Rightarrow C_2 = 0.$

$\therefore \; C_1 = 1$

$\therefore \; \boxed{y = e^{\frac{-t}{2}}}$

p-6

Q.6

(b) $y'' + 0.2y' + 4.01y = 0.$ $\quad y'(0)=2, \quad y(0)=0$

$s^2 + 0.2s + 4.01 = 0$

$$s = \frac{-0.2 \pm \sqrt{0.04 - 16.04}}{2} \qquad \left(x = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \text{ for } ax^2+bx+c=0 \right)$$

$$= \frac{-0.2 \pm \sqrt{-16}}{2}$$

$$= -0.1 \pm 4i$$

$\therefore s_1 = -0.1 + 4i \qquad s_2 = -0.1 - 4i$

$\therefore y = c_1 e^{(0.1+4i)t} + c_2 e^{(-0.1-4i)t}$

$y(t) = e^{-0.1}\left(c_1 e^{4it} + c_2 e^{-4it} \right)$

$\therefore y'(t) = e^{-0.1}\left(4ic_1 e^{4it} - 4ic_2 e^{-4it} \right)$

$y'(0) = 2 \Rightarrow 2 = 4e^{-0.1}\left(c_1 e^{4it} - c_2 e^{-4it} \right)$

$\therefore e^{0.1} = 2c_1 e^{4it} - 2c_2 e^{-4it}$

$y(0) = 0 \Rightarrow 0 = c_1 e^{4it} + c_2 e^{-4it} \Rightarrow \cancel{c_2 = -c_1 e^{8it}}$

$0 = c_1 + c_2 \Rightarrow c_2 = -c_1$

$\therefore e^{0.1} = 4c_1\left(e^{4it} - e^{-4it}\right) = 8c_1\, i \sin(4t)$

$\therefore c_1 = \frac{-e^{0.1}}{8\sin(4t)} i, \quad c_2 = -c_1$

$$= \frac{-e^{0.1+\frac{\pi}{2}i}}{8\sin(4t)}$$

$\therefore y = c_1 e^{(\ )} + c_2^{(\ )}$