In-Class Test #1 (Date - 22/08/2019)

Engg. Math. for Advanced Studies

(Module - Linear Algebra).

Autumn 2019

Solutions to Problems

Q.1

$$\left.\begin{array}{l} x+y+z=1 \\ x+y+z=3 \end{array}\right\} \rightarrow \text{These two are parallel planes in } \mathbb{R}^3$$

$x-y=1$ → This one is a plane parallel to Z-axis obtained by extruding line $x-y=1$ i.e. $y=x-1$

A) Singular.

Q.2

$x+y+z = 135000$ → Total amount received / invested

$3x+2y+z = 200000$ → After one year

$x+y = 100000$ → last piece of info. given.

a)
$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 135000 \\ 200000 \\ 100000 \end{bmatrix} \rightarrow (Ax=b).$$

b) $R_2 \rightarrow R_2 - 3R_1$ & $R_3 \rightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 135000 \\ -205000 \\ -35000 \end{bmatrix}$$

↑ Upper triangular form → $(Ux=c)$

②

c) $\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 135000 \\ 200000 \\ 100000 \end{bmatrix}$

$$E \cdot A \cdot x = E \cdot b$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$\boxed{Q.3}$

Given A in $Ax=b$ is a 5×7 matrix with rank 3.

$m=5$, $n=7$, $r=3$. mapping from $\mathbb{R}^7$ to $\mathbb{R}^5$

a) $p=7$ $q=5$

b) $\dim(N(A)) = n-r = 7-3 = 4$

c) $\dim(\cancel{CA}(A^T)) = r$

d) $\dim(C(A)) = r$

e) $\dim(N(A^T)) = m-r = 5-3 = 2$

f) Number of sprecial solutions expected is equal to number of free variables, i.e. rank subtracted from $\dim(C(A))$.

$= 7-3 = \underline{\underline{4}}$

Q.4

Typical method to check independence of given set of vectors is to put them as rows of a matrix and perform row operations to get upper triangular matrix. e.g. say,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$R_2' = R_2 - R_1$ & $R_4' = R_4 + R_2'$ gives

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$R_4'' = R_4' - R_3$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank} = 3$$

a) $r < m$ hence vectors are NOT lenearly independent.

b) $r = 3 \rightarrow$ 3 dimensional space will be spanned by given 4 vectors in $\mathbb{R}^4$

Q.5

To check if a transformation is linear or not, see if it obeys two conditions required for linearity.

if $u = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ & $v = \begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix}$ are two vectors in the space on which transformation T is applied,

$$T(u+v) = T(u) + T(v) \quad \text{--- condition \#1}$$

$$T(\alpha u) = \alpha T(u) \quad \text{--- condition \#2}$$

α being a scalar.

Check. for linearity

$$T(u+v) = T\left(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix}\right) = T\left(\begin{Bmatrix} (x_1+x_3) \\ (x_2+x_4) \end{Bmatrix}\right) = \begin{Bmatrix} x_2+x_4 \\ x_1+x_3 \end{Bmatrix}$$

$$T(u)+T(v) = T\left(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\right) + T\left(\begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix}\right) = \begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} + \begin{Bmatrix} x_4 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} x_2+x_4 \\ x_1+x_3 \end{Bmatrix}$$

Hence first condition satisfied.

$$T(\alpha u) = T\left(\alpha \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\right) = T\left(\begin{Bmatrix} \alpha x_1 \\ \alpha x_2 \end{Bmatrix}\right) = \begin{Bmatrix} \alpha x_2 \\ \alpha x_1 \end{Bmatrix}$$

$$\alpha T(u) = \alpha\left(T\left(\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}\right)\right) = \alpha\left(\begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix}\right) = \begin{Bmatrix} \alpha x_2 \\ \alpha x_1 \end{Bmatrix}$$

Hence second condition is also satisfied.

Answer : Yes, it is a linear transformation