IIT DHARWAD AUTUMN 2019

P901
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Final Exam.
Au19

Engg. Mathematics for Advanced Studies

MODULE - LINEAR ALGEBRA
FINAL EXAM - SAMPLE SOLUTION

Q.1

(a) $3x + 2y + 2z = 0$

(b) No. P does not contain $(0,0,0)$.
Any space has to have ~~to~~ $\vec{O}$.

(c) Yes. P_0 is a vector space.

Q.2

(a) Yes. $(b_1 = 0)$

(b) $(b_1 = 1)$ NO ∵ (excludes $(0,0,0)$)

(c) $b_2 * b_3 = 0$ NO ∵ (~~cos~~ consider $(0,1,0)$ & $(0,0,1)$ added to get $(0,1,1)$ which is outside $b_2 * b_3$)

(d) Yes.
if $\vec{u} = (1, 1, 0)$ & $\vec{v} = (2,0,1)$
any combination of those two will form vector space.

(e) Yes.
(It is a plane passing through origin.)

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Q.3

(a)

Say $A = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}$

& $F = \begin{bmatrix} — v_1 — \\ — v_2 — \\ \vdots \\ — v_4 — \end{bmatrix} = A^T$

To qualify as basis vectors, the vectors should be: (i) independent set of vectors (ii) should span the ~~n~~ entire space.

Since we have 4 vectors, in $\mathbb{R}^4$

$\mathbb{R}$ dim. of space $\mathbb{R}^4 = 4$ & ~~4~~ basis vectors are expected

To check independence, we perform row elimination on F.

$$\begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 2 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{of 1st column}]{\text{elim.}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2nd colm.}]{\text{Elim.}} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\Big\downarrow \text{Elim. on 3rd column}$$

$$\{v_1, v_2, v_3, v_4\} \Leftarrow \text{Rank} = 4 \Leftarrow \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

~~a~~ is set of independent vectors.

& They span $\mathbb{R}^4$

(a) $\boxed{\text{They qualify as basis vectors}}$

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$\boxed{Q.3}$ contnd...

(b) Rank of A^T is 4. (Rank of F was evaluated earlier).

Hence, determinant has to be non-zero.

Hence, pivot value of A can not be zero since product of pivot values gives the determinant.

$\boxed{\text{False — None of pivot values will be zero}}$

(c) $\det(D) = \det\left[(AB^TC)^2\right] = \left(\det(AB^TC)\right)^2$

$= \left[\det(A) \cdot \det(B^T) \cdot \det(C)\right]^2$

$= \left[\det(A) \cdot \det(B) \cdot \det(C)\right]^2$ ... $\left[\det(B) = \det(B^T)\right]$

Now, in (a) we found that pivot values of $F (= A^T)$ were 1, 1, -1, 2, Hence

$\det(A) = \det(A^T) = \det(F) = 1 \times 1 \times (-1) \times 2 = -2$

Since B is obtained by swapping 2^{nd} & 3^{rd} row of A^T

$\det(B) = -\det(A^T) = -\det(A) = 2$

... (Rule → Row swapping changes sign of determinant)

Since C is obtained by adding 2 times 3^{rd} row of A to 4^{th} row, $\det(C) = \det(A)$

$\det C = -2$ ... (Rule → Addition/subtraction of row multiple does not change determinant)

$\therefore \det(D) = \left[(-2) \cdot (2) \cdot (-2)\right]^2 = 64$

$\boxed{\text{Determinant } D = 64}$

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Q.4

(a) FALSE - rank could be smaller than $m = n$

(b) TRUE

(c) TRUE

(d) TRUE

(e) FALSE - If columns are dependent one can get $Ax = 0$ for $x \neq 0$.

Q.5

(a) Yes $\rightarrow$ if $\{v, u\} \in U$ & $\{u, v\} \in V$

so that $\{u, v\} \in U \cap V$

Any combination of u, v will always belong to to both U & V.

(b) $\dim(U) = \dim(V) = 4$

Consider u_1, u_2, u_3 & u_4 to form basis of U

& $v_1, v_2 \cdots v_4$ to form basis of V

As $U \in \mathbb{R}^7$ & $V \in \mathbb{R}^7$, set of above 8 basis vector can not be independent. $U \cap V$ will be spanned by all 8 but ~~Minimum 1 vector~~ maximum 7 of those would be independent as it is $\mathbb{R}^7$

In that case $\dim(U \cap V) = 1$

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Q.5 contd..

On the other hand $U \cap V$ has to include at most 4 dimensions as its dimension can not exceed that of U & V i.e. 4.

$$\therefore \quad \boxed{1 \leq \dim(U \cap V) \leq 4}$$

i.e. 1, 2, 3, or 4 dimensions.

Q.6

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \cancel{///}$$

Making first column Zero;

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} A \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

Making 2^{nd} column zero:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

which is upper triang.

i.e. Reversing row operations on RHS.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Further reversing previous step for making first column terms zero below diagonal element:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 20 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = LU}$$

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Q.7

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) To find x_1 in null space of A,

$$Ax_1 = 0 \Rightarrow \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} a \\ b \\ c \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$\therefore$

$$a + 2b = 0$$
$$b + 2c = 0$$

Choose $c = 1 \Rightarrow b = -2 \Rightarrow a = 4$

$$\therefore x_1 = \begin{Bmatrix} 4 \\ -2 \\ 1 \end{Bmatrix}$$

... (Note → any other choice of a, b, c which obeys above two eqns. is equally valid).

(b) $x_2 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Rightarrow x_3 = x_1 + x_2 = \begin{Bmatrix} 5 \\ -1 \\ 2 \end{Bmatrix}$

$x_3 = \begin{Bmatrix} 5 \\ -1 \\ 2 \end{Bmatrix}$ $Ax_3 = b$.

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \begin{Bmatrix} 5 \\ -1 \\ 2 \end{Bmatrix} = \begin{Bmatrix} 3 \\ 3 \end{Bmatrix}$$

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(a) If $n = m$ there is always at most one solution. (Marks 2*)

(b) If $n > m$ you can always solve $Ax = b$ (Marks 2*)

(c) If $n > m$ the nullspace of A has dimension greater than zero (Marks 2*)

(d) If $n < m$ then for some b there is no solution of $Ax = b$ (Marks 2*)

(e) If $n < m$ the only solution of $Ax = 0$ is $x = 0$ (Marks 2*)

5. Let U and V both be four-dimensional subspaces of $\mathbb{R}^7$, and let $W = U \cap V$.

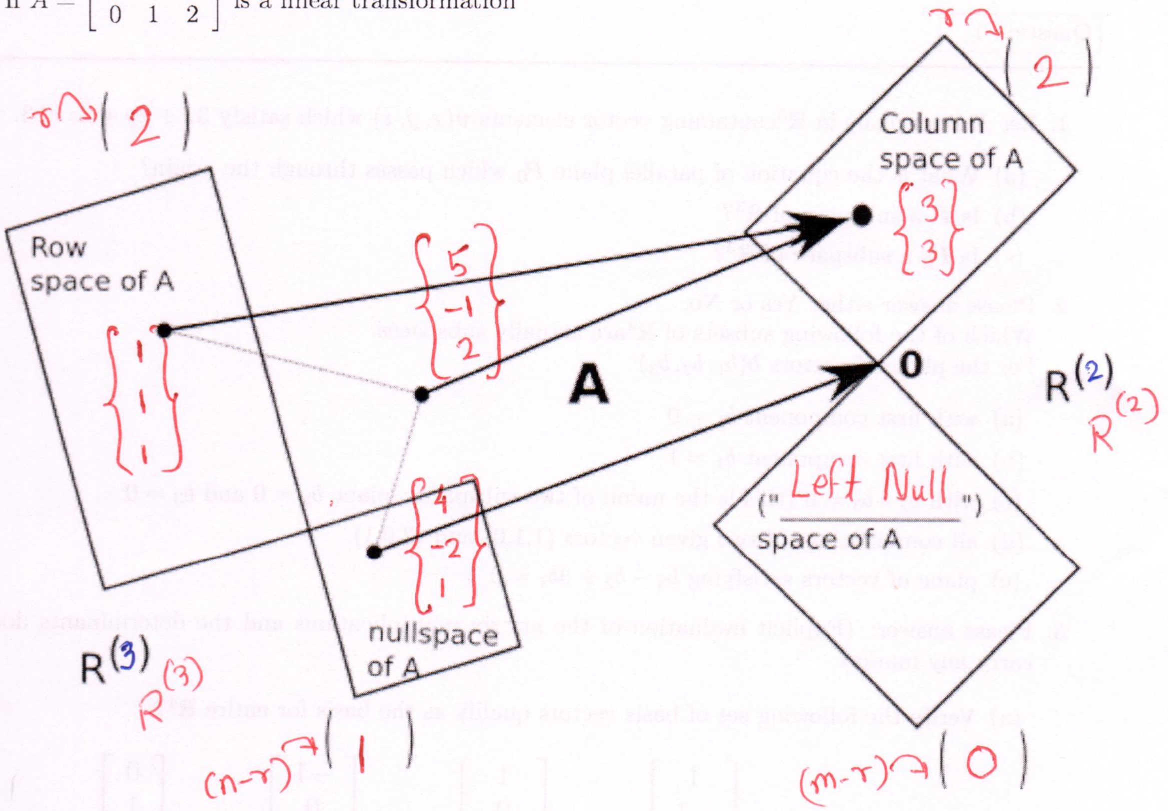
(a) Will W be always a vector space? (Yes/No) (Marks 2*)

(b) If W is a vector space, what are possible values for the dimension of W ? (Hint: Create an example to form and verify your thoughts) (Marks 2*)

6. Perform LU decomposition for matrix A given below: (Marks 3)

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

7. If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ is a linear transformation



(a) find a vector x_1 in the null space of A (Marks 4)

(b) find a new vector x_3 by adding vector $x_2 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ to the x_1. Find b such that $Ax_3 = b$. (Marks 1)

(Note - if you are not sure about actual x_1 calculation in above (a) assume it to be $x1 = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$ to proceed further)

(c) Leveraging the fact that A is a linear transformation i.e. $Ax_3 = A(x_1 + x_2)$, represent **this particular** $Ax_3 = b$ in above diagram:

i. Enter correct number in the empty brackets above R. i.e. $\mathbb{R}^{(?)}$ (two places) (Marks 1+1)

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Q. 8

(a) $\text{rank}\, A = m$

(b) $\dim(\text{Column Space } A) = m$

(c) dimension of left null space $= m - r = m - m = 0$.

Q. 9

Take center of clock as the origin & consider the two directions of the minute & hour hand as the vectors in $\mathbb{R}^2$.

After time t, starting from noon we can get respective unit vector directions (~~$\vec{u}_m$ & u~~ $\hat{e}_m$ & $\hat{e}_h$) as below

$$\hat{e}_m = \begin{Bmatrix} \cos\left(\frac{\pi}{2} - \omega_m t\right) \\ \sin\left(\frac{\pi}{2} - \omega_m t\right) \end{Bmatrix}$$

... ω_m is angular vel. of minute hand & $\theta = 0 \Rightarrow$ direction along '3' of the clock.

$$= \begin{Bmatrix} \sin(\omega_m t) \\ \cos(\omega_m t) \end{Bmatrix}$$

$$\hat{e}_h = \begin{Bmatrix} \sin(\omega_h t) \\ \cos(\omega_h t) \end{Bmatrix}$$

-- ω_h is angular vel. of hour hand.

Since, we are interested in time t_p when these two directions will be perpendicular,

$$\hat{e}_h \cdot \hat{e}_m = 0 \qquad \text{.. when } t = t_p$$

$$\therefore \sin(\omega_m t_p) \sin(\omega_h t_p) + \cos(\omega_m t_p) \cos(\omega_h t_p) = 0$$

$$\therefore \cos\left(\left(\omega_m - \omega_h\right) t\right) = 0 \quad --$$

$\boxed{Q.9}$ continued...

$$\cos\left((\omega_m - \omega_h)t_p\right) = 0$$

$$\therefore \quad (\omega_m - \omega_h)t_p = \frac{n\pi}{2} \quad \ldots \text{ where } n = 1, 2, 3 \ldots$$

Since, we know it ~~ith~~ is the first time when hands are becoming perpendicular, we choose $n = 1$.

i.e.
$$(\omega_m - \omega_h)t_p = \frac{\pi}{2} \qquad \text{———} \quad 9_a$$

Since, minute hand completes on revolution in 1 hr. i.e. 2π rotation in 3600 s.

$$\omega_m = \frac{2\pi}{3600}$$

similarly
$$\omega_h = \frac{2\pi}{12 \times 3600} \quad \ldots (2\pi \text{ in } 12 \text{ hrs}).$$

$$\therefore \quad \omega_m = 12\omega_h$$

$\therefore$ eqn 9_a becomes

$$(12\omega_h - \omega_h)t_p = \frac{\pi}{2}$$

$$\therefore \quad t_p = \frac{\pi}{(11\omega_h) \times 2} = \frac{\pi}{\left(11 \times \frac{2\pi}{12 \times 3600}\right) \times 2}$$

$$\therefore \quad \boxed{\begin{array}{l} t_p = 981.8181 \text{ s} \\ t_p = 16.363 \text{ min.} \end{array}}$$

Q.10

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Transformation is linear if

$T(x_1, x_2)$ is such that

$T(\alpha x_1 + \beta x_2) = \alpha T(x_1) + \beta T(x_2)$.

$\therefore$ ~~$T(x_1) =$~~ To avoid confusion with symbols used in this Problem, restating above as:

$T : V \to W$ for any vectors $\vec{v}_1$ & $\vec{v}_2 \in V$

$$T(\alpha \vec{v}_1 + \beta \vec{v}_2) = \alpha T(\vec{v}_1) + \beta T(\vec{v}_2)$$

Given $T(\vec{v}) = T(x_1, x_2)$ .. for $\vec{v} \in \mathbb{R}^2$ where $\vec{v} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$

$$= \sqrt{x_1^2 + x_2^2}$$

$\therefore$ If $\vec{v}_1 = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$ $\vec{v}_2 = \begin{Bmatrix} x_3 \\ x_4 \end{Bmatrix}$

$T(\vec{v}_1) = \sqrt{x_1^2 + x_2^2}$ — 10 a

$T(\vec{v}_2) = \sqrt{x_3^2 + x_4^2}$ — 10 b

$$\alpha \vec{v}_1 + \beta \vec{v}_2 = \begin{Bmatrix} \alpha x_1 + \beta x_3 \\ \alpha x_2 + \beta x_4 \end{Bmatrix}$$

$$\therefore T(\alpha \vec{v}_1 + \beta \vec{v}_2) = \sqrt{(\alpha x_1 + \beta x_3)^2 + (\alpha x_2 + \beta x_4)^2}$$

$$= \sqrt{\begin{pmatrix} \alpha^2 x_1^2 + \beta^2 x_3^2 + 2\alpha\beta x_1 x_3 \\ + \alpha^2 x_2^2 + \beta^2 x_4^2 + 2\alpha\beta x_2 x_4 \end{pmatrix}}$$

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Q.10

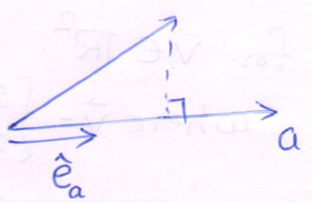
Since,

$$T(\alpha\vec{v}_1 + \beta\vec{v}_2) \neq \alpha T(\vec{v}_1) + \beta T(\vec{v}_2)$$

Given transformation is NOT linear.

$\therefore$ $\boxed{T(x_1, x_2) = \sqrt{x_1^2 + x_2^2} \quad \text{is NON-LINEAR}}$

Q.11



$$\hat{e}_a = \frac{\vec{a}}{|a|} = \frac{\vec{a}}{a^T\vec{a}}$$

Component of $\vec{b}$ along $\vec{a}$ = $(b \cdot a)\hat{e}_a$

$\therefore$ Projection of b along a i.e. $\vec{p}$ ~~vec~~

$$= (\vec{b}\cdot\vec{a})\frac{\vec{a}}{|\vec{a}|} = \frac{aa^T}{a^Ta}\vec{b} \qquad \text{—— (11a)}$$

For $a = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad aa^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$$a^Ta = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 1$$

$\therefore$ $\vec{p} = \left(\frac{1}{1}\right)\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ ... (putting $\vec{b}$ & aa^T a^Ta in (11a))

$\boxed{\text{(a)} \quad \vec{p} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix}}$ $\qquad$ $\boxed{\text{(b)} \quad \mathbb{P} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}}$

Q.12

Inner product remains unchanged under orthogonal transformation.

$\therefore \quad \vec{u}_1 \cdot \vec{u}_2 = (Q\vec{u}_1) \cdot (Q\vec{u}_2) = \vec{v}_1 \cdot \vec{v}_2$

$\therefore$ as $\vec{u}_1 = \begin{bmatrix} 3 \\ 2 \\ -6 \end{bmatrix}$ & $\vec{u}_2 = \begin{bmatrix} -3 \\ 4 \\ -6 \end{bmatrix}$

$\vec{v}_1 \cdot \vec{v}_2 = \vec{u}_1 \cdot \vec{u}_2 = 35$

Hence, $\boxed{\text{Answer (b) \& (c)}}$

Q.13

(a) TRUE → there are 2 eigen vectors & there is no repeated eigen value. Hence, diagonalization $A = S \Lambda S^{-1}$ is possible.

(b) FALSE → zero eigen value implies singularity i.e. non-invertibility

(c) 0 (zero) → zero eigen value ⇒ determinant of A is 0.

(d) if $\vec{v}_1$ & $\vec{v}_2$ are eigen vectors & λ_1 & λ_2 respective eigen values, of A

$A = [S][\Lambda][S^{-1}]$ where $S = \begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix}$

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Q.13

$$S = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Say $S^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

For first colm on RHS:

$$\therefore \left.\begin{aligned} a + 2c &= 1 \\ 2a - c &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} a &= \frac{1}{5} \\ \& \ c &= \frac{2}{5} \end{aligned}$$

For second column on RHS

$$\left.\begin{aligned} b + 2d &= 0 \\ 2b - d &= 1 \end{aligned}\right\} \Rightarrow \begin{aligned} b &= 2/5 \\ d &= -1/5 \end{aligned}$$

$$\therefore S^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

(Note → One can use Gauss Jorden method as well to get inverse)

$$\therefore A = S\Lambda S^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}\left(\frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\right)$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 0 \\ 10 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$$

$$\therefore \boxed{A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}}$$

(e) $A = S\Lambda S^{-1}$

$A^k = S\Lambda^k S^{-1}$

$\boxed{Q.13}$ continued.

(e) $A^3 = S\Lambda^3 S^{-1}$

$$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 0 \end{bmatrix}^3 \begin{bmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 125 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 125 & 0 \\ 250 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 125 & 250 \\ 250 & 500 \end{bmatrix}$$

$$\boxed{A^3 = \begin{bmatrix} 25 & 50 \\ 50 & 100 \end{bmatrix}}$$

$\boxed{Q.14}$

(a) Hermitian matrix means conjugate transposes gives back original matrix.

$$A = \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix} \quad \& \quad A^H = \begin{bmatrix} 1 & \overline{1+i} \\ \overline{1-i} & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1-i \\ 1+i & 2 \end{bmatrix}$$

$$= A.$$

Hence, $\boxed{A = A^H \Rightarrow A \text{ is Hermitian.}}$

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Q.14

(b)

$$B = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

Unitary matrix U, satisfy ~~B^H~~ $U^H U = I$

$$B^H = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Rightarrow B^H B = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$$

$$\therefore B^H B = \begin{bmatrix} 2 & 2i \\ -2i & 2 \end{bmatrix} \neq I$$

Hence B is NOT unitary matrix

Q.15

Answer : (c) No pivot is negative

(e) $x \cdot Ax \geqslant 0$ for any vector x.

EACH of Above conditions are necessary and sufficient conditions for semi-positive definiteness of matrix

Q. 16

(a) $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix}$ i.e. $A_{2\times 3}$

SVD :

$$A = U \Sigma V^T$$

$U \rightarrow$ columns of U are eigen vectors of AA^T

$V \rightarrow$ columns of V are eigen vectors of A^TA

$\Sigma \rightarrow$ $m\times n$ matrix if A is $A_{m\times n}$. (diagonal elements are square root of eigen values of AA^T & A^TA)

$$A_{2\times 3} = [U]_{2\times 2} [\Sigma]_{2\times 3} [V^T]_{3\times 3}$$

(b) $A = U\Sigma V^T = u_1\sigma_1 v_1^T + u_2\sigma_2 v_2^T + \dots + u_r \sigma_r v_r^T$

If $\sqrt{\lambda_1^2 + \lambda_8^2 + \lambda_{10}^2 + \lambda_{14}^2}$ contains 95% of $\sqrt{\Sigma \lambda_i^2}$

Only those four eigen vectors can approximate matrix 'A' reliably using corresponding eigen vectors of AA^T & A^TA

$$\therefore \quad A \approx u_1\sigma_1 v_1^T + u_8\sigma_8 v_8^T + u_{10}\sigma_{10}v_{10}^T + u_{14}\sigma_{14}v_{14}^T$$

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Q.17

Since, $\vec{u}_i$ $i = 1$ to n are also vectors in $\mathbb{R}^n$ ~~sp~~ spanned by $\vec{v}_i$ $i = 1$ to n, each of those can be expressed as linear combination of $\vec{v}_i$'s i.e.

$$\vec{u}_1 = (\vec{u}_1 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_1 \cdot \vec{v}_2)\vec{v}_2 + (\vec{u}_1 \cdot \vec{v}_3)\vec{v}_3 \cdots + (\vec{u}_i \cdot \vec{v}_n)\vec{v}_n$$

$$\vec{u}_2 = (\vec{u}_2 \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_2 \cdot \vec{v}_2)\vec{v}_2 + \cdots + (\vec{u}_2 \cdot \vec{v}_n)\vec{v}_n$$

$$\vdots$$

$$\vec{u}_n = (\vec{u}_n \cdot \vec{v}_1)\vec{v}_1 + (\vec{u}_n \cdot \vec{v}_2)\vec{v}_2 + \cdots + (\vec{u}_n \cdot \vec{v}_n)\vec{v}_n$$

$$\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} (u_1 \cdot v_1) & \cdots & (u_n \cdot v_1) \\ (u_1 \cdot v_2) & & (u_n \cdot v_2) \\ (u_1 \cdot v_3) & A & \vdots \\ \vdots & & \vdots \\ (u_1 \cdot v_n) & & (u_n \cdot v_n) \end{bmatrix}$$

(Note - due to ambiguity in wording this question, full marks are given to all the students)