IIT DHARWAD

AUTUMN 2019

# Engg. Mathematics For Advanced Studies.

MODULE-LINEAR ALGEBRA

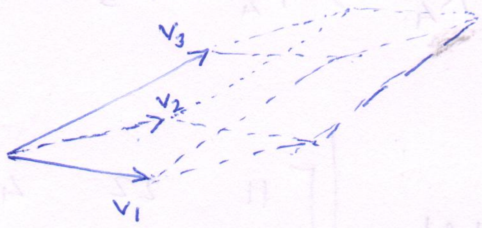
# ASSIGNMENT - 04 - SOLUTIONS

## Q.1

$\vec{u}_1 = -\hat{i} - \hat{j} + 2\hat{k}$

$\vec{u}_2 = \hat{i} + 2\hat{k}$

$\vec{u}_3 = \hat{j} + 2\hat{k}$




Volume of parallelopiped = det A where $A = \begin{bmatrix} - u_1^T - \\ - u_2^T - \\ - u_3^T - \end{bmatrix}$

(will it be same if $A = \begin{bmatrix} | & | & | \\ u_1 & u_2 & u_3 \\ | & | & | \end{bmatrix}$ ? food for thought!).

$$\text{Volume} = \begin{vmatrix} -1 & -1 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ -1 & -1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{vmatrix}$$

$(R_1 \leftrightarrow R_2,\ R_2 \leftrightarrow R_3)$ $\qquad (R_3 \rightarrow R_3 + R_1)$

$$= \begin{vmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{vmatrix} = 6 \text{ units.}$$

$R_3 \longrightarrow R_3 + R_2$

$$\boxed{\text{Volume} = 6 \text{ cubic units}}$$

②

$\boxed{Q.2}$

Lets say
$$A = \begin{bmatrix} 11 & 22 & 44 & 22 \\ 0 & 20 & 40 & 200 \\ 0 & 0 & 25 & 50 \\ 0 & 0 & 50 & 108 \end{bmatrix}$$

with $R_4 \rightarrow R_4 - 2R_3$ ... (Rule → addition/subtraction of a 'k' times row does not change detereminant).

$$|A| = \left|\begin{bmatrix} 11 & 22 & 44 & 22 \\ 0 & 20 & 40 & 200 \\ 0 & 0 & 25 & 50 \\ 0 & 0 & 0 & 8 \end{bmatrix}\right|$$

$$= 11 \times \left|\begin{bmatrix} 1 & 2 & 4 & 2 \\ 0 & 20 & 40 & 200 \\ 0 & 0 & 25 & 50 \\ 0 & 0 & 0 & 8 \end{bmatrix}\right|$$

(Rule → $\det\begin{vmatrix} ac & bc \\ f & g \end{vmatrix} = a \times \det\begin{vmatrix} a & b \\ f & g \end{vmatrix}$)

$$= 11 \times 20 \times 25 \times 8 \times \det\begin{vmatrix} 1 & 2 & 4 & 2 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$ ... (repeating above for each row)

$$= 11 \times 20 \times 25 \times 8 \times \left(1 \times 1 \times 1 \times 1\right)$$ ... (Rule → detereminant of a triangular matrix is equal to product of diagonal elements)

$$= 44000$$

Hence, $\det(A^{-1}) = \dfrac{1}{\det(A)} = \dfrac{1}{44000}$ — (Rule → $\det(A^{-1}) = \dfrac{1}{\det A}$)

Answer: $\boxed{\det A^{-1} = \dfrac{1}{44000}}$

# Q.3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\left( \because \text{ Rule: } \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \begin{vmatrix} a & 0 \\ c & d \end{vmatrix} + \begin{vmatrix} 0 & b \\ c & d \end{vmatrix} \right)$$

OR in other simplified way of writing:

$$= \begin{vmatrix} a_{11} & 0 & 0 \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix} + \begin{vmatrix} 0 & a_{12} & 0 \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix} + \begin{vmatrix} 0 & 0 & a_{13} \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix}$$

$$= \underbrace{a_{11} \begin{vmatrix} 1 & 0 & 0 \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix}}_{\text{say } A^{11}} + \underbrace{a_{12} \begin{vmatrix} 0 & 1 & 0 \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix}}_{A^{12}} + \underbrace{a_{13} \begin{vmatrix} 0 & 0 & 1 \\ - & A_{2k} & - \\ - & A_{3k} & - \end{vmatrix}}_{A^{13}}$$

$$\left( \text{Rule: } \det \begin{vmatrix} ac & bc \\ f & g \end{vmatrix} = c \times \det \begin{vmatrix} a & b \\ f & g \end{vmatrix} \right)$$

$=$

First term $A^{11}$

$$A^{11} = a_{11} \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \times \left[ \begin{vmatrix} 1 & 0 & 0 \\ a_{21} & 0 & 0 \\ - & A_{3k} & - \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & a_{22} & 0 \\ - & A_{3k} & - \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & a_{23} \\ - & A_{3k} & - \end{vmatrix} \right]$$

$$= a_{11} \times \left( \underbrace{\cancel{a_{21} \begin{vmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ - & A_{3k} & - \end{vmatrix}}}_{=0 \text{ as two rows are identical.}} + a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ - & A_{3k} & - \end{vmatrix} + a_{23} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ - & A_{3k} & - \end{vmatrix} \right)$$

$$= a_{11} \times \left( a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \right)$$

④

# $\boxed{Q.3}$ continued

$$A_{11} = a_{11} \times \left( a_{22} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \right)$$

$$= a_{11} \times \left( a_{22} \times \left( \cancel{a_{31}} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & a_{33} \end{vmatrix} \right) + a_{23} \times \left( \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ a_{31} & 0 & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & a_{32} & 0 \end{vmatrix} + \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a_{33} \end{vmatrix} \right) \right)$$

$$= a_{11} \times \left( a_{22} \times \left( a_{31} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix}}_{=0} + a_{32} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix}}_{=0} + a_{33} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}}_{=1} \right) + a_{23} \times \left( a_{31} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{vmatrix}}_{=0} + a_{32} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}}_{=-1} + a_{33} \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{vmatrix}}_{=0} \right) \right)$$

( −1 comes because Row exchange gives identity & one row exchange means det. sign flips).

$$= a_{11} \times \left( a_{22} \times \left( a_{33} \right) + a_{23} \times (-a_{32}) \right)$$

$$A^{11}_{\cancel{11}} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

similarly $A^{12}_{\cancel{22}} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$ & $A^{13}_{\cancel{3}} = a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Hence,

$$\det \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}.$$

# $\boxed{Q.\ 4}$

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$$

ⓐ $A\vec{x} = \vec{u} \Rightarrow \vec{x} \in \mathbb{R}^3$ & $\vec{u} \in \mathbb{R}^2$

Hence, $p = 3$ & $q = 2$

ⓑ Maximize $A\vec{x} \Rightarrow$ Maximizing $\|A\vec{x}\| \Rightarrow$ Maximizing $\|A\vec{x}\|^2$

However $\|A\vec{x}\|^2 = (A\vec{x}) \cdot (A\vec{x}) = (A\vec{x})^T (A\vec{x}) = x^T A^T A x$ —— Ⓐ

Say $B = A^T A$.

$B$ is symmetric positive semi definite matrix. because for any $\vec{y} \in \mathbb{R}^3$ $\vec{y} \cdot B\vec{y} = \vec{y} \cdot (A^T A \vec{y}) = y^T A^T A y = (A\vec{y}) \cdot (A\vec{y})$

$= \|A\vec{y}\| \geq 0$

Going back to Ⓐ above,

$x^T A^T A x = x^T B x = x^T S \Lambda S^T x$ — where $S$ is eigen vector matrix & $\Lambda$ is diagonal matrix of $\lambda$. i.e. eigen values.

(assuming $B$ has all independent eigen vectors).

$S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$ where $Bv_i = \lambda_i v_i$

$= \tilde{x}^T \Lambda \tilde{x}$ —— Ⓑ

$B = S \Lambda S^T \rightarrow$ (spectral decomposition)

where $\tilde{x} = S^T x$. is transformation of $x$ to Eigen vector basis.

( Observe : $S^T x = \begin{bmatrix} — v_1^T — \\ — v_2^T — \\ — v_n^T — \end{bmatrix} \begin{bmatrix} x \end{bmatrix} = \begin{bmatrix} \vec{v}_1 \cdot \vec{x} \\ \vec{v}_2 \cdot \vec{x} \\ \vec{v}_n \cdot x \end{bmatrix}$

i.e. each row on RHS is a scalar value equal to projection of vector $\vec{x}$ on ~~$v_i$~~ ith eigen vector $\vec{v}_i$. Hence ~~as~~ if $\|v_i\| = 1$, $\|S^T x\| = 1$ )

(6)

Going back to (B) again,

$$\|A\vec{x}\|^2 = \cancel{\tilde{x}^T B \tilde{x}} \; [\tilde{x}]^T [\Lambda][\tilde{x}]$$

$$= [\tilde{x}]^T \begin{bmatrix} \lambda_1 \tilde{x}_1 \\ \lambda_2 \tilde{x}_2 \\ \vdots \\ \lambda_i \tilde{x}_i \\ \vdots \\ \lambda_n \tilde{x}_n \end{bmatrix}$$

$$= \lambda_1 \tilde{x}_1^2 + \lambda_2 \tilde{x}_2^2 + \cdots + \lambda_n \tilde{x}_n^2 \quad \text{— (C)}$$

Given the fact that i) all $\lambda_i \geq 0$

ii) $\|\vec{x}\| = 1 \Rightarrow \sum_{i=1}^{n} x_i^2 = 1$

as well as $\sum_{i=1}^{n} \tilde{x}_i^2 = 1$

using ii) as the constrain, to maximize in (C), we should have $\tilde{x}_i = 1$ for $\lambda_i = \lambda_{max}$ & as ~~a result~~ an obvious implication $\tilde{x}_i = 0$ for $\lambda_i \neq \lambda_{max}$

i.e $(S^T\vec{x})_i = 1$ i.e. $\begin{bmatrix} \vec{v}_1 \cdot \vec{x} \; (=0) \\ \vec{v}_2 \cdot \vec{x} \; (=0) \\ \vdots \\ \vec{v}_i \cdot \vec{x}_i \; (=0) \\ \vdots \\ \vec{v}_p \cdot \vec{x}_p (=1) \\ \vdots \\ \vec{v}_n \cdot \vec{x}_n \; (=0) \end{bmatrix}$ if $\lambda_p$ is $\lambda_{max}$

i.e. $\vec{x}$ is along $\vec{v}_p$ which is eigen vector corresponding ~~i.e~~ to maximum eigen value $\lambda_p$.

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When input $\vec{x}_{max}$ is along eigen vector of the $A^TA$ with max eigenvalue, we have maximized $\|A\vec{x}\|$.
Similar argument can be made for minimization of $\|A\vec{x}\|$ to find $\vec{x}_{min}$ along eigen vector of $A^TA$ with Minimum eigen value.

# Q.4 contnd.

(c) Observe ~~s~~ (c) eqn. given previously along with the ~~two~~ an implications of $A^T A$ being symmetric & constrain that $\|\vec{x}\| = 1$ given after eqn. (c)

To minimize we should have

$\tilde{x}_i = 1$ for $\lambda_i = \lambda_{min}$

& $\tilde{x}_i = 0$ for all other i's. because $\|\tilde{x}\| = \|\vec{x}\| = 1$

Hence, $(S^T \vec{x})_i = 1 \Rightarrow \vec{v}_q \cdot \vec{x} = 1$ for $q$ such that $\lambda_q = \lambda_{min}$

i.e. $\vec{x}$ should be along (eigen vector of B) $\hat{\vec{v}}_q$ which has minimum $\lambda_q = \lambda_{min}$.

(d) For $A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}$ (GNU octave commands will follow as an appendix).

$$B = A^T A = \begin{bmatrix} 80 & 100 & 40 \\ 100 & 170 & 140 \\ 40 & 140 & 200 \end{bmatrix}$$

S, which is matrix containing eigen vectors of B & L which is diagonal matrix consisting eigen values in the same sequence as those of eigen vectors given in S, is given below.

$$S = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix} \quad L = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 90 & 0 \\ 0 & 0 & 360 \end{bmatrix}$$

# Q.4 (continued)

i.e. in increasing order of $\lambda_i$

$\lambda_1 = 0$ $\quad v_1 = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$

$\lambda_2 = 90$ $\quad v_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

$\lambda_3 = 360$ $\quad v_3 = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

Maximize $\|A\vec{x}\|$

Hence, to maximize $\|Ax\|$, $x = v_3$ as $\lambda_3 = \lambda_{max} = 360$

(as $\|v_3\| = 1$ we need no scaling)

$\therefore \vec{x}_{max} = \vec{v}_3$

$$\vec{x}_{max} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$$

To Minimize $\|A\vec{x}\|$

$\vec{x} = \vec{v}_1$ as $\lambda_1 = \lambda_{min} = 0$.

(as $\|v_1\| = 1$, we need no scaling)

$\vec{x}_{min} = \vec{v}_1$

$$\vec{x}_{min} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$$

# Q.4 continued:

(e) for finding where the minimized & maximized vectors $A\vec{x}_{max}$ & $A\vec{x}_{min}$ lie in $\mathbb{R}^2$.

$$\vec{p} = A\vec{x}_{max} = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix} = \begin{bmatrix} 18 \\ 6 \end{bmatrix}$$

$$\vec{r} = A\vec{x}_{min} = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

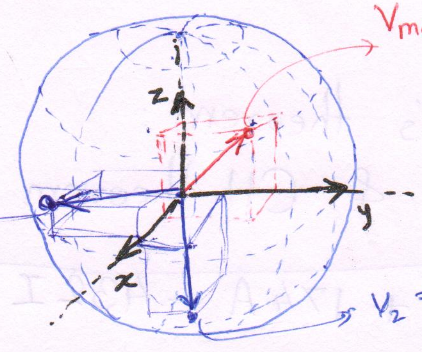
Aside $\Rightarrow$ for $v_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$. $A\vec{x} = A\vec{v}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

say. $\vec{q} = A\vec{v}_2 = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$.

(f) Input space surface $\rightarrow$ $\vec{x} \in \mathbb{R}^3$ with $\|\vec{x}\| = 1 \Rightarrow$ sphere in 3D.

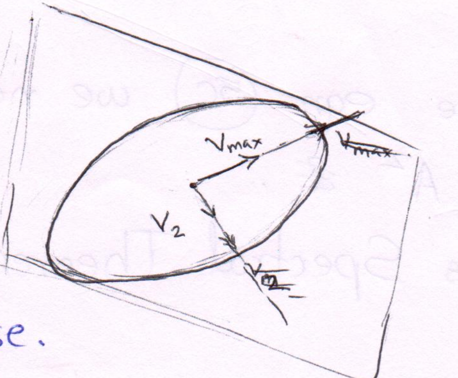
$V_{max} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

$V_{min} = \begin{bmatrix} 2/3 \\ -2/3 \\ 1/3 \end{bmatrix}$




$V_2 = \begin{bmatrix} 2/3 \\ 1/3 \\ -2/3 \end{bmatrix}$

Output space is 2D space spanned by $\vec{V}_{max}$ & $\vec{V}_2$ & all points from 3D input sphere maps on an ellipse.




NOTE $\rightarrow$ in a way "A" is projecting a sphere on an 2D ellipse.
observe $\lambda_1 = 0$
Also observe why A is NOT invertible

10

# Q.5

(a) $A = \begin{bmatrix} 8 & 0 & 1 \\ 0 & 8 & 1 \\ 1 & 1 & 7 \end{bmatrix}$

Characteristic Equation : $|A - \lambda I| = 0$

$$\begin{vmatrix} (8-\lambda) & 0 & 1 \\ 0 & (8-\lambda) & 1 \\ 1 & 1 & (7-\lambda) \end{vmatrix} = 0$$

$$(8-\lambda)\begin{vmatrix} (8-\lambda) & 1 \\ 1 & (7-\lambda) \end{vmatrix} + (1)\begin{vmatrix} 0 & 8-\lambda \\ 1 & 1 \end{vmatrix} = 0$$

$$(8-\lambda)(54 - 15\lambda + \lambda^2) = 0 \quad \text{—— eqn } 5A$$

$$432 - 174\lambda + 23\lambda^2 - \lambda^3 = 0$$

$$\boxed{\lambda^3 - 23\lambda^2 + 174\lambda - 432 = 0.}$$ characteristic equation. = eqn 5B

from eqn. 5A,

$$(8-\lambda)(\lambda - 9)(\lambda - 6) = 0.$$

i.e. $\boxed{\lambda_1 = 6, \quad \lambda_2 = 8, \quad \lambda_3 = 9.}$

(b) Cayley-Hamilton's theorem :

Using eqn. 5B & CH theorem

$$\boxed{A^3 - 23A^2 + 174A - 432I = 0.} \quad \text{—— eqn } 5C$$

(c) To verify above eqn. 5C we need to calculate $A^3$, $A^2$ ~~&~~.

We leverage ~~us~~ Spectral Theorem for that.

# Q.5 continued,

(c) $A = S \Lambda S^{-1}$ provided $A$ has enough distinct eigen vectors $\vec{v}_i$ that form $S$.

$$S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}.$$

Using GNU octave, we find .. (GNU Octave) Command: $[S, L] = \text{eig}(A)$

$$S = \begin{bmatrix} -0.4082 & -0.7071 & -0.5774 \\ -0.4082 & 0.7071 & -0.5774 \\ 0.8165 & 0 & -0.5774 \end{bmatrix}$$

$$\Lambda = L = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^3 = S \Lambda^3 S^{-1}$$

However, observing that $S \cdot S^T = I$ above, $S^{-1} = S^T$

$$\therefore A^3 = S \Lambda^3 S^T = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} 6^3 & 0 & 0 \\ 0 & 8^3 & 0 \\ 0 & 0 & 9^3 \end{bmatrix} \begin{bmatrix} S^T \end{bmatrix} = \begin{bmatrix} 535 & 23 & 171 \\ 23 & 535 & 171 \\ 171 & 171 & 387 \end{bmatrix}$$

$$A^2 = S \Lambda^2 S^T = \begin{bmatrix} S \end{bmatrix} \begin{bmatrix} 6^2 & 0 & 0 \\ 0 & 8^2 & 0 \\ 0 & 0 & 9^2 \end{bmatrix} \begin{bmatrix} S^T \end{bmatrix} = \begin{bmatrix} 65 & 1 & 15 \\ 1 & 65 & 15 \\ 15 & 15 & 51 \end{bmatrix}$$

Substituting $A^3$ & $A^2$ in eqn. (5C) we can verify that Cayley-Hamilton Theorem is applicable e Multiplication of pivots = Multiplication of eigen values

( = determinant.)

Hence Multiplication of pivot = M

$$\boxed{M = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 6 \times 8 \times 9 = 432}$$

f as all eigen values $\lambda_i > 0$
$(\lambda_1 = 6,\ \lambda_2 = 8,\ \lambda_3 = 9)$

$\boxed{\text{A is Positive Definite Matrix}}$

```
GNU Octave, version 4.2.2
>> version
ans = 4.2.2


>>% ********************************************
>>% QUESTION 4 - Assignment04 LinAlgebra
>>% ********************************************

>> A=[4 8; 11 7; 14 -2]
A =
    4   8
   11   7
   14  -2

>> B=A'*A
B =

    80  100   40
   100  170  140
    40  140  200

>> [S,L]=eig(B)
S =

   0.66667   0.66667   0.33333
  -0.66667   0.33333   0.66667
   0.33333  -0.66667   0.66667

L =

Diagonal Matrix

  -2.3143e-14           0           0
            0  9.0000e+01           0
            0           0  3.6000e+02

>> S*S'
ans =

   1.0000e+00   1.5420e-16   1.3569e-16
   1.5420e-16   1.0000e+00   1.9737e-16
   1.3569e-16   1.9737e-16   1.0000e+00
```

```
>> % COMMENT : Observe above that the V is an ORTHOGONAL matrix as the column vectors are ORTHONORMAL.
>> % COMMENT : A Quick verification for Spectral theorem follow in next command.
>> S*L*(S')
ans =

   80.000   100.000    40.000
  100.000   170.000   140.000
   40.000   140.000   200.000

>> B
B =

   80   100    40
  100   170   140
   40   140   200

>> COMMENT :  Another quick check for eigen vectors
>> B*S(:,1)
ans =

  -1.7764e-15
   7.1054e-15
   4.2633e-14

>> B*S(:,2)
ans =

   60.000
   30.000
  -60.000

>> B*S(:,3)
ans =

   120.00
   240.00
   240.00

>> COMMENT :  Verify the norm of the eigen vector is one or not
>> norm(S(:,1))
ans =  1.00000
```

```
>> norm(A*S(:,3))
ans =  18.974



>>% ********************************************
>>% QUESTION 5 - Assignment04 LinAlgebra
>>% ********************************************

>> A2=[8 0 1; 0 8 1; 1 1 7]
A2 =

   8   0   1
   0   8   1
   1   1   7

>> C=A2
C =

   8   0   1
   0   8   1
   1   1   7

>> eig(C)
ans =

   6.0000
   8.0000
   9.0000
>> % COMMENT:  Before we start, Cayley Hamilton verification brute force
method!
>> C*C*C - 23*C*C +174*C-432*eye(3)
ans =

0   0   0
0   0   0
0   0   0


>> [S2, L2] = eig(A2)
S2 =

  -4.0825e-01  -7.0711e-01  -5.7735e-01
```

```
  -4.0825e-01   7.0711e-01  -5.7735e-01
   8.1650e-01   1.1483e-15  -5.7735e-01

L2 =

Diagonal Matrix

   6.0000        0        0
        0   8.0000        0
        0        0   9.0000

>> % COMMENT : Before using Spectral Theorem, let compute A^3
>> A2*A2*A2
ans =

   535    23   171
    23   535   171
   171   171   387

>> % Using Spectral theorem:
>> S2*(L2^3)*S2'
ans =

   535.000    23.000   171.000
    23.000   535.000   171.000
   171.000   171.000   387.000
```