IIT DHARWAD. AUTUMN 2019

ENGG MATHS FOR ADVANCED STUDIES

MODULE - LINEAR ALGEBRA

ASSIGNMENT 03 SOLUTIONS

Q.1

Orthogonal basis → set of vectors which are:
i) orthogonal to each other.
ii) which span entire vector space.

Orthonormal basis → set of vectors
i) which are orthogonal to each other
ii) which span entire vector space
iii) each ~~ev~~ vector is of unit magnitude.

Q.2

Orthogonal matrix → i) Square matrix
ii) All column vectors form orthonormal set of vectors.

By def^n, $QQ^T = I$ (and also $Q^TQ = I$). Hence, $Q^{-1} = Q^T$ i.e. transpose the orthogonal matrix to get inverse.

Q.3

Informational Review of Gram-Schmidt process.

Q.4

(a) $A = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 3 & -1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow \vec{V}_{a1} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V}_{a2} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \vec{V}_{a3} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

First basis vector:

$\vec{q}_1 = \dfrac{\vec{V}_{a1}}{\|\vec{V}_{a1}\|} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ ⟶ First basis vector using Gram-Schmidt orthogonalization

Second basis vector:

$\vec{q}_1 \cdot \vec{V}_{a2} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} = \dfrac{8}{\sqrt{6}}$

②

$$\vec{q}_2' = \vec{V}_{c2} - (\vec{q}_1 \cdot \vec{V}_{c2})\vec{q}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \frac{8}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) = \frac{+}{2}$$

$$= \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \frac{8}{6}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 - \frac{8}{3} \\ 3 - 4/3 \\ 1 - 4/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$$

$$\therefore \vec{q}_2 = \frac{\vec{q}_2'}{\|\vec{q}_2'\|} = \frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}$$ $\longrightarrow$ Second basis vector using Gram-Schmidt.

(Note: We can simply ignore the common factor $1/3$ in $\vec{q}_2'$ to simplify calculations).

Third basis vector:

$$\vec{q}_2 \cdot \vec{V}_{c3} = \left(\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) = \frac{-5}{\sqrt{30}}$$

$$\vec{q}_1 \cdot \vec{V}_{c3} = \left(\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right) = \frac{-1}{\sqrt{6}}$$

$$\vec{q}_3' = \vec{V}_{c3} - (\vec{q}_1 \cdot \vec{V}_{c3})\vec{q}_1 - (\vec{q}_2 \cdot \vec{V}_{c3})\vec{q}_2$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} - \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}\right) - \left(\frac{-5}{\sqrt{30}}\right)\left(\frac{1}{\sqrt{30}}\begin{bmatrix} -2 \\ 5 \\ -1 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1/3 \\ 1/6 \\ 1/6 \end{bmatrix} + \begin{bmatrix} -1/3 \\ 5/6 \\ -1/6 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{i.e. } \vec{q}_3' = \vec{0} \Rightarrow \vec{q}_3 = \vec{0}$$ $\longrightarrow$ Third basis vector as per Gram-Schmidt orthogonalization.

$$\vec{q}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{0}$$

Note $\rightarrow$ Only 2 basis vectors, namely $\vec{q}_1$ & $\vec{q}_2$ span the column space of $A_{3\times3}$ matrix.

$$B = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 1 & 1 \end{bmatrix} \quad \vec{V}_{b1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \vec{V}_{b2} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \vec{V}_{b3} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

First basis Vector:

$$\vec{q}_1 = \frac{\vec{V}_{b1}}{\|\vec{V}_{b1}\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \longrightarrow$$ First basis vector as per Gram-Schmidt orthogonalization.

Second basis Vector:

$$\vec{q}_1 \cdot \vec{V}_{b2} = \left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}\right) = 2\sqrt{3}$$

$$\vec{q}_2' = \vec{V}_{b2} - (\vec{q}_1 \cdot \vec{V}_{b2})\vec{q}_1 = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} - \frac{(2\sqrt{3})}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{q}_2'}{\|\vec{q}_2'\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \rightarrow$$ Second basis vector as per Gram-Schmidt orthogonalization.

Third basis vector:

$$\vec{q}_1 \cdot \vec{V}_{b3} = \left(\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \frac{4}{\sqrt{3}}$$

$$\vec{q}_2 \cdot \vec{V}_{b3} = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{2}}$$

$$\vec{q}_3' = \vec{V}_{b3} - (\vec{q}_1 \cdot \vec{V}_{b3})\vec{q}_1 - (\vec{q}_2 \cdot \vec{V}_{b3})\vec{q}_2$$

$$= \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{3}\cdot\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \frac{1}{\sqrt{2}\cdot\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/3 \\ 4/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} 0 \\ 1/2 \\ -1/2 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 6-8+0 \\ 12-8-3 \\ 6-8+3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{q}_3 = \frac{\vec{q}_3'}{\|\vec{q}_3'\|} = \frac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \longrightarrow$$ Third basis vector as per Gram-Schmidt orthogonalization.

④

$$|A| = 2\begin{vmatrix} 3 & -1 \\ 1 & 0 \end{vmatrix} - 2\begin{vmatrix} 1 & -1 \\ 1 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 2 + 0 = 0$$

$$|B| = 1\begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} + 1\begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} = 1 + 2 - 2 = 1$$

Comment :-

A matrix is singular as its determinant is zero.
It means that the columns are not independent.
(side note : $\det(A^T) = \det(A)$ hence
$\det(A) = 0 \Rightarrow \det(A^T) = 0 \Rightarrow$ rows of A^T are not independent.

This fact is again reflected (as expected) ~~s~~ in the ~~Q~~ Gram-Schmidth orthogonalization where only $\vec{q}_1$ & $\vec{q}_2$ are enough to span the column space defined by 3 column vectors of A as ~~&~~ $\vec{q}_3 = \vec{0}$. i.e. all 3 columns vectors of A are in 2D plane in $\mathbb{R}^3$ defined by $\vec{q}_1$ & $\vec{q}_2$.
~~In~~

In case of matrix B, determinant of B is non-zero i.e. 3 columns vectors are independent. i.e. dimension of column space is expected to be 3.
We find that 3 basis vectors $\vec{q}_1, \vec{q}_2$ & $\vec{q}_3$ are obtained by Gram-schmidt orthogonalization.

$\boxed{Q\cdot 5}$

$$C = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \quad \vec{V}_{c1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{V}_{c2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \vec{V}_{c3} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

First basis vector as per Gram-Schmidt orthogonalization:

$$\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \ldots \left(\because \vec{q}_1 = \frac{\vec{V}_{c1}}{\|\vec{V}_{c1}\|}\right) \longrightarrow \text{First basis vector}$$

Second basis vector:

$$\vec{q}_1 \cdot \vec{V}_{c2} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}$$

$$\vec{q}'_2 = \vec{V}_{c2} - (\vec{q}_1 \cdot \vec{V}_{c2})\vec{q}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{2}\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{q}'_2}{\|\vec{q}'_2\|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \longrightarrow \text{Second basis vector.}$$

Third basis vector:

$$\vec{q}_1 \cdot \vec{V}_{c3} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \sqrt{2}$$

$$\vec{q}_2 \cdot \vec{V}_{c3} = \left(\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right) \cdot \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right) = \sqrt{2}$$

$$\vec{q}'_3 = \vec{V}_{c3} - (\vec{q}_1 \cdot \vec{V}_{c3})\vec{q}_1 - (\vec{q}_2 \cdot \vec{V}_{c3})\vec{q}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{q}_3 = \frac{\vec{q}'_3}{\|\vec{q}'_3\|} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \longrightarrow \text{Third Basis vector.}$$

(a) $$Q = \begin{bmatrix} | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{bmatrix}$$

quick dot product of 1st & 2nd row vector is Zero.
Proof that for all orthogonal matrices, ~~all~~ row vectors are orthogonal ~~follows~~ (and orthonormal) follows from $QQ^T = I$.

(6)

~~$QQ^T = I$~~

$$QQ^T = I \Rightarrow \begin{bmatrix} \multimap & \multimap & \cdots & \multimap \\ \leftarrow & \leftarrow & \cdots & \leftarrow \\ \star & \star & \cdots & \star \\ q_1 & q_2 & & q_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} \multimap & \uparrow & \star & q_1 & \cdots \\ \multimap & \uparrow & \star & q_2 & \cdots \\ \vdots & & & & \\ \vdots & & & & \\ \multimap & \uparrow & \star & q_n & \cdots \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

Considering ~~Comparing~~ any I_{ij} with $i \neq j$ we see any two different rows are orthogonal.

(b)
$$\left.\begin{aligned} \vec{V}_{c1} &= \sqrt{2}\,\vec{q}_1 \\ \vec{V}_{c2} &= \frac{1}{\sqrt{2}}\vec{q}_1 + \frac{1}{\sqrt{2}}\vec{q}_2 \\ \vec{V}_{c3} &= \sqrt{2}\,\vec{q}_1 + \sqrt{2}\,\vec{q}_2 + \vec{q}_3 \end{aligned}\right\} \Rightarrow C = \begin{bmatrix} | & | & | \\ \vec{V}_{c1} & \vec{V}_{c2} & \vec{V}_{c3} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ q_1 & q_2 & q_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1/\sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}$$

In the above matrix decomposition of C we find that first matrix is orthogonal & the second one is Upper triangular Matrix

(c) It is obvious for any matrix as long as $q_i \neq \vec{0}$
↪ (upper triangular matrix).