IIT DHARWAD.

AUTUMN 2019

ENGG. MATHS FOR ADVANCED STUDIES

ASSIGNMENT 02 SOLUTIONS.

MODULE - LINEAR ALGEBRA

Q.1

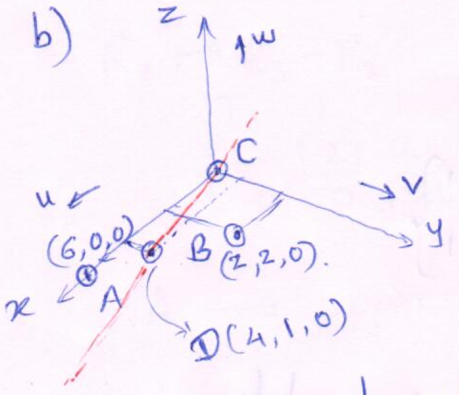
Given reference plane $u+2v-w=6$.

a) parallel plane through origin:

$$u+2v-w=0$$

(Note: any plane $u+2v-w=$ const. is in family of parallel planes to given ref. plane. RHS constant is decided using a point on the plane)

b)



Two methods to decide this plane:

(i) by observation, we can see that all 3 points A,B,C are on a x-y plane passing through origin.

So, eqn. of the plane:

$$z \quad \boxed{w=0}$$

(ii) more general method uses cross product of two vectors (which are derived from the 3 points given).

$$\vec{v}_1 = \vec{A}-\vec{B} = (4,-2,0)^T$$

$$\vec{v}_2 = \vec{A}-\vec{C} = (6,0,0)^T$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -2 & 0 \\ 6 & 0 & 0 \end{vmatrix} = -12\hat{k} \longrightarrow$$

$w=$ constant.

this defines family of planes only. One of the points on plane will exactly define the plane. Origin C gives e.g. $\boxed{w=0}$

c) Point D $(4,1,0)$ is also marked above.

Plane is $\perp^{er}$ to x-y plane. Line CD (i.e. $x-4y=0$) is extruded along $\vec{z}$ to form this plane.

$$\boxed{x-4y+0z=0}$$

②

Q.2

$$\left.\begin{aligned} u+v+w&=0 \quad \text{--- (2a)}\\ u+2v+3w&=0 \quad \text{--- (2b)}\\ 3u+5v+7w&=1 \quad \text{--- (2c)}. \end{aligned}\right\} \begin{bmatrix}1&1&1\\1&2&3\\3&5&7\end{bmatrix}\begin{Bmatrix}u\\v\\w\end{Bmatrix}=\begin{Bmatrix}0\\0\\1\end{Bmatrix}$$

By Gaussian Elimination we have

$$\begin{matrix} R_2\to R_2-\cancel{2}R_1 \\ R_3\to R_3-3R_1\end{matrix} \quad \begin{bmatrix}1&1&1\\0&1&2\\0&2&4\end{bmatrix}\begin{Bmatrix}u\\v\\w\end{Bmatrix}=\begin{Bmatrix}0\\0\\1\end{Bmatrix} \underset{R_3\to R_3-2R_2}{\Rightarrow} \begin{bmatrix}1&1&1\\0&1&2\\ \boxed{0\;\;0\;\;0}\end{bmatrix}\begin{Bmatrix}u\\v\\w\end{Bmatrix}=\begin{Bmatrix}0\\0\\ \boxed{1}\end{Bmatrix}$$

As observed in the last equation, this is an in-consistent system of algebraic equations & Hence, no solutions

Q.3

$$\left.\begin{aligned} kx+y&=1\\ x+ky&=1\end{aligned}\right\} \Rightarrow \begin{bmatrix}k&1\\1&k\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}=\begin{Bmatrix}1\\1\end{Bmatrix}.$$

$$R_1 \leftrightarrow R_2 \Rightarrow \begin{bmatrix}1&k\\k&1\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}=\begin{Bmatrix}1\\1\end{Bmatrix}$$

$$R_2\to R_2-kR_1 \Rightarrow \begin{bmatrix}1&k\\0&1-k^2\end{bmatrix}\begin{Bmatrix}x\\y\end{Bmatrix}=\begin{Bmatrix}1\\1-k\end{Bmatrix}$$

a) When $1-k^2=0$ but $1-k\neq 0 \Rightarrow$ No solution.
$\therefore (1-k)(1+k)=0 \Rightarrow \boxed{1+k=0 \Rightarrow} \boxed{K=-1 \Rightarrow \text{No. Solution Condition}}$

b) When the determinant $\begin{vmatrix}1&k\\0&1-k^2\end{vmatrix}\neq 0$, we have unique solution.
$\therefore 1-k^2\neq 0 \Rightarrow \boxed{K\neq 1 \text{ and } K\neq -1 \Rightarrow \text{~~No~~ Unique Sol}^n}$

c) When $K=1 \Rightarrow$ determinant is zero as well as rhs second element is zero $\Rightarrow$ Rank=1 with both eqns being identical for

$\boxed{k=1 \text{ we have infinity of solutions}}$

Q.4

(a) $$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$A \cdot x = b$$

(b) $|A| = 1(15-9) - 1(5-3) + 1(3-3) = 4$

(c) $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 2 \\ 0 & 2 & 4 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & \boxed{2} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}$$

$A \longrightarrow$ upper triangular $A' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix}$

(d) Pivot values as shown above are $\rightarrow 1, 2, 2$

$1 \times 2 \times 2 = 4$ — (note $\Rightarrow$ same as $|A|$ calculated in (b).)

(e) Lets say with given modification in last eqn. We have $Bx = b$

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & 3 \\ 2 & 3 & 3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 2R_1]{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - \frac{1}{2} R_2} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 2 \\ 0 & 0 & \boxed{0} \end{bmatrix}$$

$|B| = 1(3\times3 - 3\times3) - 1(1\times3 - 3\times2) + 1(1\times3 - 3\times2) = 0$

Pivots : 1, 2, 0

Product of pivot $= 1 \times 2 \times 0 = 0$

4

Q.5

$$A = \begin{bmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & 0 & 1 \end{bmatrix}$$

Gauss Jordan method $\Rightarrow$ $[A \mid I] \xrightarrow[\text{Operations}]{\text{Row}} [I \mid A^{-1}]$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ p & 1 & 0 & 0 & 1 & 0 \\ q & 0 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 \to R_3 - qR_1]{R_2 \to R_2 - pR_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -p & 1 & 0 \\ 0 & 0 & 1 & -q & 0 & 1 \end{array}\right]$$

Hence $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -p & 1 & 0 \\ -q & 0 & 1 \end{bmatrix}$

Q6

a) $LHS = 0\vec{u} = (0+0)\vec{u} = 0\vec{u} + 0\vec{u} \rightarrow (\because 0 = 0+0$ Distributive law of

$\therefore 0\vec{u} = 2(0\vec{u})$ $\longrightarrow$ this is equivalent to $\vec{v} = 2\vec{v}$ or $\vec{v} = \alpha\vec{v}$

~~Given that $\vec{u} \neq 0$ always.~~ Only possible if $\vec{v} = 0$ ~~or $\vec{u}$~~

$\therefore 0\vec{u} = \vec{0}$

b) $LHS = \alpha\vec{0} = \alpha(\vec{0} + \vec{0}) = \alpha\vec{0} + \alpha\vec{0}$ $(\because \vec{0} = \vec{0} + \vec{0}$ using def^n of $\vec{0}$ that $\vec{w} + \vec{0} = \vec{w}$. & Distributive law)

$\alpha\vec{0} = 2(\alpha\vec{0})$

Hence, either $\alpha = 0$ and/or $\alpha\vec{0} = \vec{0}$.

$\alpha\vec{0} = \vec{0} = RHS$ — (proved)

c) For each vector $\vec{u}$ in vector space there exists unique $-\vec{u}$ such that $\vec{u} + (-\vec{u}) = \vec{0}$ hence $\vec{0}$ is also linear combination of $\vec{u}$.

Hence $\vec{0}$ should be part of each vector space.

(Also $\vec{u} + \vec{0} = \vec{u} \Rightarrow \vec{u} - (\vec{u}) = \vec{0}$ $\rightarrow$ Another rationale)

$\boxed{Q:7}$

Verify if all conditions for Addition & Multiplication are obeyed by [3×4] matrix if $A_{3\times4}$ & $B_{3\times4}$ are two elements of this proposed vector space, & $C_{3\times4}$

(I) Addition

(1) Addition is commutative. (?)

$$A_{3\times4} + B_{3\times4} \overset{?}{=} B_{3\times4} + A_{3\times4} \longrightarrow \text{verified} \checkmark$$

(2) Addition is associative (?)

$$\left(A_{3\times4} + B_{3\times4}\right) + C_{3\times4} \overset{?}{=} A_{3\times4} + \left(B_{3\times4} + C_{3\times4}\right)$$

$\longrightarrow$ verified ✓

(3) There exist unique vector $\vec{O}_{3\times4}$ such that (?)

$$A_{3\times4} + O_{3\times4} \overset{?}{=} A_{3\times4} \longrightarrow \text{verified} \checkmark$$

$$\vec{O}_{3\times4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(4) There exists unique vector $-A_{3\times4}$ such that (?)

$$A_{3\times4} + \left(-A_{3\times4}\right) \overset{?}{=} \vec{O}_{3\times4} \longrightarrow \text{verified} \checkmark$$

$$-A_{3\times4} = \begin{bmatrix} -a_{11} & \cdots & -a_{1n} \\ -a_{21} & & \vdots \\ -a_{31} & \cdots & -a_{nn} \end{bmatrix}$$

(II) Multiplication by scalar α, β.

(1) Associative (?) $\alpha\left(\beta A_{3\times4}\right) = (\alpha\beta) A_{3\times4}$

$\longrightarrow$ verified ✓

(2) $(1) \cdot A_{3\times4} \overset{?}{=} A_{3\times4}$ (?) $\longrightarrow$ verified ✓

III 1. Multiplication by a scalar is distributive with respect to vector addition

$$\alpha\left(A_{3\times4} + B_{3\times4}\right) \stackrel{?}{=} \alpha A_{3\times4} + \alpha B_{3\times4} \quad (?)$$

$\longrightarrow$ verified ✓

2. Multiplication by a ~~vect~~ scalar is distributive with respect to scalar addition

$$(\alpha+\beta)\, A_{3\times4} \stackrel{?}{=} \alpha A_{3\times4} + \beta A_{3\times4}$$

$\longrightarrow$ verified ✓

As all above conditions are satisfied for any $A_{3\times4}$ in the vector space and any scalar α in scalar field, Proposed vector space of [3x4] matrices is Verified as vector space

Q.8

Positive quadrant of x-y plane is NOT a vector space as with negative α, & β we can show that $(\alpha\vec{v}_1 + \beta\vec{v}_2)$ is not in positive quadrant.

e.g. $\alpha=-1$ & $\beta=0$

Q.9

Any example like billing in hotel for a group of customer, refilling fuel in diff. fuel stations along the to & fro journey where individual fuel stations fuel may have been compromized. etc...